Online Supplement to "Resource Allocation via Message Passing"

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A. Proofs of Existence Theorems

In the following, a function $f\colon \mathcal{S} \to \mathbb{R}$ with domain $\mathcal{S} \subset \mathbb{R}^n$ is said to be Lipschitz with constant L if

$$|f(x) - f(y)| \leq L\|x - y\|, \quad \forall\, x, y \in \mathcal{S}.$$

Theorem 1. *Assume that the utility functions are Lipschitz continuous. Then, a message-passing equilibrium exists.*

Proof. Let L be a Lipschitz constant that applies to all utility functions. Suppose each message in the set V is Lipschitz continuous with Lipschitz constant L. Consider the message from an activity a to a resource $r \in \mathcal{R}(a)$. Define $\mathcal{X}^{a\backslash r} \triangleq \prod_{r \in \mathcal{R}(a)\backslash r} \mathcal{X}_r$ to be the space of consumption bundles for activity a, excluding resource r. Without loss of generality, assume that $(FV)_{a\to r}(x'_{ar}) \geq (FV)_{a\to r}(x_{ar})$. Then, for some $z' \in \mathcal{X}^{a\backslash r}$,

$$\begin{aligned}(FV)_{a\to r}(x'_{ar}) - (FV)_{a\to r}(x_{ar}) &= u_a(x'_{ar}, z') + \sum_{r' \in \mathcal{R}(a)\backslash r} V_{r'\to a}(z'_{ar'}) \\ &\quad - \max_{z \in \mathcal{X}^{a\backslash r}} \left(u_a(x_{ar}, z) + \sum_{r' \in \mathcal{R}(a)\backslash r} V_{r'\to a}(z_{ar'}) \right) \\ &\leq u_a(x'_{ar}, z') - u_a(x_{ar}, z') \leq L|x'_{ar} - x_{ar}|.\end{aligned}$$

Hence, the message $(FV)_{a\to r}(\cdot)$ is Lipschitz continuous with Lipschitz constant L. A similar proof applies to $(FV)_{r\to a}(\cdot)$.

Let $\mathcal{S}$ be the collection of message sets V for which each message equals zero at zero and is Lipschitz continuous with Lipschitz constant L. Note that $\mathcal{S}$ is convex, closed, and bounded (under the supremum norm). $\mathcal{S}$ is subset of the set of continuous functions from a compact, finite dimensional metric space to itself. Hence, $\mathcal{S}$ is compact under the supremum norm by the Arzelà-Ascoli theorem. The operator H maps $\mathcal{S}$ to $\mathcal{S}$ continuously with respect to the supremum norm. It follows from the Schauder fixed point theorem that a message-passing equilibrium exists. $\square$

Theorem 3. *There exists a message-passing equilibrium with concave and Lipschitz continuous messages.*

Proof. The proof follows by a modification of the proof of Theorem 1: define the set $\mathcal{S}'$ to be the collection of message sets $V \in \mathcal{S}$ which are also concave. Since the operator H involves maximization of a concave function over a convex set, if $V \in \mathcal{S}'$, then HV is also concave hence $HV \in \mathcal{S}'$. The existence of a fixed-point in $\mathcal{S}'$ follows from the Schauder fixed point theorem. $\square$

B. Proofs of Optimality Theorems

We start with two preliminary lemmas.

Lemma 1. *Given a message-passing equilibrium V and an allocation decision x^*, the following three conditions are equivalent:*

(i) For every activity a, the allocation $x^*_{\mathcal{R}(a)}$ uniquely maximizes the activity manager's problem

$$\text{(B.1)} \qquad \begin{array}{lll} \text{maximize} & U_a(x_{\mathcal{R}(a)}) \triangleq & u_a(x_{\mathcal{R}(a)}) + \sum_{r\in\mathcal{R}(a)} V_{r\to a}(x_{ar}) \\ \text{subject to} & & x_{ar} \in \mathcal{X}_r, \qquad \forall\, r \in \mathcal{R}(a). \end{array}$$

(ii) For every resource r, the allocation $x^*_{\mathcal{A}(r)}$ uniquely maximizes the optimization problem

$$\text{(B.2)} \qquad \begin{array}{lll} \text{maximize} & U_r(x_{\mathcal{A}(r)}) \triangleq & \sum_{a\in\mathcal{A}(r)} V_{a\to r}(x_{ar}) \\ \text{subject to} & & \sum_{a'\in\mathcal{A}(r)} x_{a'r} \le b_r, \\ & & x_{a'r} \in \mathcal{X}_r, \qquad \forall\, a' \in \mathcal{A}(r). \end{array}$$

(iii) For every activity a and every resource $r \in \mathcal{R}(a)$, the quantity x^*_{ar} uniquely maximizes the optimization problem

$$\text{(B.3)} \qquad \begin{array}{lll} \text{maximize} & U_{ar}(x_{ar}) \triangleq & V_{a\to r}(x_{ar}) + V_{r\to a}(x_{ar}) \\ \text{subject to} & & x_{ar} \in \mathcal{X}_r. \end{array}$$

Proof. Given an activity a and a resource $r \in \mathcal{R}(a)$, define

$$\mathcal{C}_{a\to r} \triangleq \left\{ x_{\mathcal{R}(a)\setminus r} \ : \ x_{ar'} \in \mathcal{X}_r,\ \forall r' \in \mathcal{R}(a) \setminus r \right\}.$$

This is the set of consumption decisions of activity a for all resources except r. Given a resource r and an activity $a \in \mathcal{A}(r)$, define $\mathcal{C}_{r\to a}(x_{ar}) \triangleq \{x_{\mathcal{A}(r)\setminus a} \ : \ \sum_{a'\in\mathcal{A}(r)\setminus a} x_{a'r} \leq b_r - x_{ar},\ x_{a'r} \in \mathcal{X}_r,\ \forall a' \in \mathcal{A}(r) \setminus a\}$. This is the set of set of feasible allocations of resource r for all activities except a, given the allocation x_{ar} to activity a. Finally, for each resource r, define $\mathcal{C}_r(x_{ar}) \triangleq \{x_{\mathcal{A}(r)\setminus a} \ : \ \sum_{a'\in\mathcal{A}(r)\setminus a} x_{a'r} \leq b_r - x_{ar},\ x_{a'r} \in \mathcal{X}_r,\ \forall a' \in \mathcal{A}(r) \setminus a\}$.

Then, from the equilibrium equation $HV = V$, we have for every x_{ar},

$$\text{(B.4)} \qquad \begin{aligned} \max_{x_{\mathcal{R}(a)\setminus r}\in\mathcal{C}_{a\to r}} U_a(x_{\mathcal{R}(a)}) &= U_{ar}(x_{ar}) + (FV)_{a\to r}(0), \\ \max_{x_{\mathcal{A}(r)\setminus a}\in C_{r\to a}(x_{ar})} U_r(x_{\mathcal{A}(r)}) &= U_{ar}(x_{ar}) + (FV)_{r\to a}(0). \end{aligned}$$

Assume that (iii) holds. Then, each $U_{ar}(\cdot)$ is maximized uniquely by x^*_{ar}. Consider an alternative feasible allocation x' with $x'_{ar} \neq x^*_{ar}$, for some activity a and resource $r \in \mathcal{R}(a)$. By (B.4), $x'_{\mathcal{R}(a)}$ cannot maximize $U_a(\cdot)$ and $x'_{\mathcal{A}(r)}$ cannot maximize $U_r(\cdot)$, respectively. Hence, (iii) implies (i) and (ii). The rest of the implications are shown similarly. □

Lemma 2. *Consider a message-passing equilibrium $HV = V$, where each activity manager's problem* (B.1) *has a unique solution, and denote the resulting allocation by x^*. Then, for each activity a and resource $r \in \mathcal{R}(a)$, this allocation maximizes the optimization problems*

$$\text{(B.5a)} \qquad \begin{aligned} &\text{maximize} && T_{r\to a}(x_{\mathcal{A}(r)}) \triangleq \textstyle\sum_{a'\in\mathcal{A}(r)\setminus a} V_{a'\to r}(x_{a'r}) - V_{r\to a}(x_{ar}) \\ &\text{subject to} && \textstyle\sum_{a'\in\mathcal{A}(r)} x_{a'r} \leq b_r, \\ &&& x_{a'r} \in \mathcal{X}_r, \quad \forall\ a' \in \mathcal{A}(r), \end{aligned}$$

$$\text{(B.5b)} \qquad \begin{aligned} &\text{maximize} && T_{a\to r}(x_{\mathcal{R}(a)}) \triangleq u_a(x_{\mathcal{R}(a)}) + \textstyle\sum_{r'\in\mathcal{R}(a)\setminus r} V_{r'\to a}(x_{ar'}) - V_{a\to r}(x_{ar}) \\ &\text{subject to} && x_{ar'} \in \mathcal{X}_{r'}, \quad \forall\ r' \in \mathcal{R}(a). \end{aligned}$$

Proof. Note that $T_{r\to a}(x_{\mathcal{A}(r)}) = U_r(x_{\mathcal{A}(r)}) - U_{ar}(x_{ar})$ and $T_{a\to r}(x_{\mathcal{R}(a)}) = U_a(x_{\mathcal{R}(a)}) - U_{ar}(x_{ar})$. The result then follows from (B.4) and Lemma 1. □

Consider a message-passing equilibrium V, assume that each activity manager's problem (B.1) has a unique solution, and define x^* to be the resulting allocation. Consider an alternative feasible allocation $x \in \mathcal{X}$. These allocations differ according to the set of transfers $\Delta(x, x^*)$. We can define sets $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{R}}$ of, respectively, activities and resources affected by the transfers by $\tilde{\mathcal{A}} = \{a \in \mathcal{A} \ : \ \exists\ r \in \mathcal{R} \text{ with } x_{ar} \neq x^*_{ar}\}$ and $\tilde{\mathcal{R}} = \{r \in \mathcal{R} \ : \ \exists\ a \in \mathcal{A} \text{ with } x_{ar} \neq x^*_{ar}\}$. Note that we have suppressed the dependence of the sets $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{R}}$ on

x and x^* for notational simplicity. We have the following theorem, from which Theorem 2 follows as an immediate corollary.

Theorem 6. *Define an undirected bipartite graph with vertices $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{R}}$, and with edges according to the set of transfers $\Delta(x, x^*)$. Then:*

(i) If the bipartite graph contains at most one cycle per connected component, then $U(x^*) \geq U(x)$.

(ii) If, in addition, the graph contains a connected component that does not have a cycle, $U(x^*) > U(x)$.

Proof. Recall the objective functions $U_a(\cdot)$, $U_r(\cdot)$, and $U_{ar}(\cdot)$ defined by the equilibrium V through the optimization problems (B.1), (B.2), and (B.3), respectively. The system objective $U(\cdot)$ can be written as

$$U(x) = \sum_{a \in \mathcal{A}} U_a(x_{\mathcal{R}(a)}) + \sum_{r \in \mathcal{R}} U_r(x_{\mathcal{A}(r)}) - \sum_{a \in \mathcal{A}} \sum_{r \in \mathcal{R}(a)} U_{ar}(x_{ar}).$$

We have the decomposition

$$\begin{aligned} U(x^*) - U(x) = & \sum_{a \in \tilde{\mathcal{A}}} \left[U_a(x^*_{\mathcal{R}(a)}) - U_a(x_p a) \right] + \sum_{r \in \tilde{\mathcal{R}}} \left[U_r(x^*_{\mathcal{A}(r)}) - U_r(x_{\mathcal{A}(r)}) \right] \\ & - \sum_{(a,r) \in \Delta(x,x^*)} \left[U_{ar}(x^*_{ar}) - U_{ar}(x_{ar}) \right]. \end{aligned}$$

By the hypothesis of the theorem, we can associate each edge $(a, r) \in \Delta(x, x^*)$ in the bipartite graph with either the vertex $a \in \tilde{\mathcal{A}}$ or the vertex $r \in \tilde{\mathcal{R}}$, in a way such that each vertex is associated with at most a single edge. Then,

$$\begin{aligned} U(x^*) - U(x) = & \sum_{a \in \tilde{\mathcal{A}}_1} \left[U_a(x^*_{\mathcal{R}(a)}) - U_{a\sigma(a)}(x^*_{a\sigma(a)}) - \left(U_a(x_{\mathcal{R}(a)}) - U_{a\sigma(a)}(x_{a\sigma(a)}) \right) \right] \\ & + \sum_{r \in \tilde{\mathcal{R}}_1} \left[U_r(x^*_{\mathcal{A}(r)}) - U_{\tau(r)r}(x^*_{\tau(r)r}) - \left(U_r(x_{\mathcal{A}(r)}) - U_{\tau(r)r}(x_{\tau(r)r}) \right) \right] \\ & + \sum_{a \in \tilde{\mathcal{A}} \setminus \tilde{\mathcal{A}}_1} \left[U_a(x^*_{\mathcal{R}(a)}) - U_a(x_{\mathcal{R}(a)}) \right] + \sum_{r \in \tilde{\mathcal{R}} \setminus \tilde{\mathcal{R}}_1} \left[U_r(x^*_{\mathcal{A}(r)}) - U_r(x_{\mathcal{A}(r)}) \right], \end{aligned}$$

where $\tilde{\mathcal{A}}_1 \subset \tilde{\mathcal{A}}$ and $\tilde{\mathcal{R}}_1 \subset \tilde{\mathcal{R}}$ are sets of vertices which have been associated with edges, and the maps $\sigma \colon \tilde{\mathcal{A}}_1 \to \tilde{\mathcal{R}}$ and $\tau \colon \tilde{\mathcal{R}}_1 \to \tilde{\mathcal{A}}$ define the associations. Observe that, by the unique optimality assumption and Lemmas 1 and 2, $U_r(x^*_{\mathcal{A}(r)}) > U_r(x_{\mathcal{A}(r)})$, $U_a(x^*_{\mathcal{R}(a)}) > U_a(x_{\mathcal{R}(a)})$, $U_r(x^*_j) - U_{ar}(x^*_{ar}) \geq U_r(x_j) - U_{ar}(x_{ar})$, and $U_a(x^*_{\mathcal{R}(a)}) - U_{ar}(x^*_{ar}) \geq U_a(x_{\mathcal{R}(a)}) - U_{ar}(x_{ar})$. Thus $U(x^*) \geq U(x)$. Under the additional assumption of Part (ii), the sets $\tilde{\mathcal{A}} \setminus \tilde{\mathcal{A}}_1$ and $\tilde{\mathcal{R}} \setminus \tilde{\mathcal{R}}_1$ cannot both be empty. Hence, $U(x^*) > U(x)$. □

Theorem 4. *Consider a message-passing equilibrium with concave and Lipschitz continuous messages. The resulting allocation of resources is globally optimal for the system manager's problem.*

Proof. Consider a message-passing equilibrium V with concave and Lipschitz continuous messages, and let x^* be the associated allocation. Assume that x^* lies in the interior of the domain of $U(\cdot)$. By (Rockafellar, 1970, Theorem 27.4), for each resource r and activity a, there must exist a supergradient $d^{ar} \in \partial u_a(x^*_{\mathcal{R}(a)})$ so that we have the first order conditions for the optimization problem (B.5b),

(B.6a)

$$d^{ar}_{ar} - \frac{d^+}{dx_{ar}} V_{a\to r}(x^*_{ar}) \le 0, \qquad d^{ar}_{ar} - \frac{d^-}{dx_{ar}} V_{a\to r}(x^*_{ar}) \ge 0,$$

(B.6b)

$$d^{ar}_{ar'} + \frac{d^+}{dx_{ar'}} V_{r'\to a}(x^*_{ar'}) \le 0, \ \forall\, r' \in \mathcal{R}(a) \setminus r, \quad d^{ar}_{ar'} - \frac{d^-}{dx_{ar'}} V_{r'\to a}(x^*_{ar'}) \ge 0, \ \forall\, r' \in \mathcal{R}(a) \setminus r.$$

Similarly, let $\lambda^*_{ar} \ge 0$ be a shadow price to the optimization problem (B.5a). Then,

(B.7a)

$$-\frac{d^+}{dx_{ar}} V_{r\to a}(x^*_{ar}) - \lambda^*_{ar} \le 0, \qquad -\frac{d^-}{dx_{ar}} V_{r\to a}(x^*_{ar}) - \lambda^*_{ar} \ge 0,$$

(B.7b)

$$\frac{d^+}{dx_{a'r}} V_{a'\to r}(x^*_{a'r}) - \lambda^*_{ar} \le 0, \ \forall\, a' \in \mathcal{A}(r) \setminus a, \quad \frac{d^-}{dx_{ar}} V_{a'\to r}(x^*_{a'r}) - \lambda^*_{ar} \ge 0, \ \forall\, a' \in \mathcal{A}(r) \setminus a.$$

Then, by (B.6a) and (B.7a),

$$\frac{d^-}{dx_{ar}} V_{a\to r}(x^*_{ar}) \le d^{ar}_{ar} \le \frac{d^+}{dx_{ar}} V_{a\to r}(x^*_{ar}), \qquad \frac{d^-}{dx_{ar}} V_{r\to a}(x^*_{ar}) \le -\lambda^*_{ar} \le -\frac{d^+}{dx_{ar}} V_{r\to a}(x^*_{ar}).$$

By concavity of $V_{a\to r}(\cdot)$ and $V_{r\to a}(\cdot)$,

$$\text{(B.8)} \qquad \frac{d}{dx_{ar}} V_{a\to r}(x^*_{ar}) = d^{ar}_{ar}, \qquad \frac{d}{dx_{ar}} V_{r\to a}(x^*_{ar}) = -\lambda^*_{ar},$$

where the derivatives must exist since the directional derivatives are equal. By (B.7b), and (B.8), we have $\lambda^*_{ar} = d^{a'r}_{a'r}$, for all $a' \in \mathcal{A}(r) \setminus a$. Then, must have $\lambda^*_{ar} = p^*_r$, for some vector $p^* \in \mathbb{R}^{\mathcal{R}}_+$, and, using (B.6b), also $d^{ar}_{ar'} = p^*_{r'}$, for all $r' \in \mathcal{R}(a) \setminus r$.

Define the vector dU by $(dU)_{ar} = p^*_r$, for each $a \in \mathcal{A}$ and $r \in \mathcal{R}(a)$. Then, $dU \in \partial U(x^*)$ is a supergradient of $U(\cdot)$ at x^*, the vector p^* is a shadow price vector for the system manager's optimization problem, and the allocation x^* is globally optimal. The case where x^* is on the boundary of the domain of $U(\cdot)$ is handled similarly. □

Theorem 5. *Let x^* be the globally optimal allocation for the system manager's problem and let p^* be a supporting price vector. Suppose that $U(\cdot)$ is differentiable at x^*. Consider a message-passing equilibrium V with concave and Lipschitz continuous messages. Then, for each activity a and resource r,*

$$\frac{d}{dx_{ar}} V_{a\to r}(x^*_{ar}) = p^*_r, \quad \frac{d}{dx_{ar}} V_{r\to a}(x^*_{ar}) = -p^*_r,$$

where the existence of the above derivatives is guaranteed. Thus,

$$\frac{\partial}{\partial x_{ar}} u_a(x^*_{\mathcal{R}(a)}) = \frac{d}{dx_{ar}} V_{a\to r}(x^*_{ar}) = -\frac{d}{dx_{ar}} V_{r\to a}(x^*_{ar}) = p^*_r.$$

Proof. This follows by the same argument as in Theorem 4, and the fact that if $U(\cdot)$ is differentiable at x^*, $\partial U(x^*) = \{\nabla U(x^*)\}$. □

References

Rockafellar, R. T. 1970. *Convex Analysis*. Princeton University Press, Princeton, NJ.