Loss-Versus-Fair: Efficiency of Dutch Auctions on Blockchains*

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Abstract

Milionis et al. [2023] studied the rate at which automated market makers leak value to arbitrageurs when block times are discrete and follow a Poisson process, and where the risky asset price follows a geometric Brownian motion. We extend their model to analyze another popular mechanism in decentralized finance for onchain trading: Dutch auctions. We compute the expected losses that a seller incurs to arbitrageurs and expected time-to-fill for Dutch auctions as a function of starting price, volatility, decay rate, and average interblock time. We also extend the analysis to gradual Dutch auctions, a variation on Dutch auctions for selling tokens over time at a continuous rate. We use these models to explore the tradeoff between speed of execution and quality of execution, which could help inform practitioners in setting parameters for starting price and decay rate on Dutch auctions, or help platform designers determine performance parameters like block times.

1. Introduction

Two of the most popular mechanisms for smart contracts to trade tokens are automated market makers (AMMs) — in which the price is determined by the contract's reserves — and Dutch auctions — in which the price is determined by the current time.

When block times are discrete, both of these mechanisms leak some value to arbitrageurs. Milionis et al. [2023] studied the rate of this value leakage for AMMs, which is closely related to the concept of "loss-versus-rebalancing," or LVR [Milionis et al., 2022]. We apply a similar analysis to Dutch auctions, deriving a closed form for their "loss-versus-fair" (LVF) — the expected loss to the seller relative to selling their asset at its contemporaneous fair price — as well as their expected time-to-fill. We also do a similar analysis for gradual Dutch auctions, a variation on Dutch auctions that supports selling tokens at a constant rate over an extended period of time.

We hope this analysis can help inform practitioners in parameterizing these auctions (e.g., choosing the initial price and decay rate) to trade off execution quality with speed of execution, as

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well as helping spur research on how to design variants of Dutch auctions that are more resistant to LVF.

Dutch auctions. Also known as descending price auctions, Dutch auctions are auctions in which an item is listed at a high price that is gradually decreased over time until a bidder accepts.

Dutch auctions are a commonly-used mechanism in blockchain-based applications, thanks to their simplicity and efficiency in an environment with high transaction costs, limited privacy, and pseudonymous identities. Unlike ascending-price or sealed bid auctions, Dutch auctions typically require only one transaction after they start — the winning bid, with price as a function of the block number. This means that failed bidders typically do not need to pay transaction fees such as "gas" for their bids or leak any information about their intents.[1] Similarly, the seller needs only consider the single, winning bid, resulting in a significant reduction in communication and computation complexity versus other auction formats such as first- or second-price auctions. Dutch auctions are also strongly shill-proof [Komo et al., 2024]: the seller has no incentive to submit any fake or shill bids to change the auction outcome.

For these reasons, Dutch auctions have been used for a variety of applications in decentralized finance:

- Liquidations in peer-to-pool lending protocols like Maker [MakerDAO, 2022] or Ajna [Patel et al., 2023]
- Rolling loans and discovering interest rates in peer-to-peer lending protocols like Blend [Galaga et al., 2023]
- Routing trades in protocols like UniswapX [Adams et al., 2023] and 1inch Fusion [1inch, 2022]
- Collecting and converting fees in protocols like Euler [Euler, 2024]

Dutch auctions can be used both for price discovery of highly illiquid assets like NFTs, and for automated execution between liquid assets. Here, we focus on the latter case, and specifically on auctions between highly liquid but volatile pairs of tokens, such as between ETH and stablecoins like USDC. In particular, we assume a *common value* setting where all potential buyers agree on the value of the asset being sold at any point in time (because, for example, the asset may be liquidly traded in other off-chain markets), and assume the price of the asset obeys *geometric Brownian motion.*

Gradual Dutch auctions. Gradual Dutch auctions (GDAs) are a variation on Dutch auctions that were introduced by Frankie et al. [2022] as a mechanism for selling NFTs or tokens at a constant target rate over an extended time period. "Continuous gradual Dutch auctions" (CGDAs), the kind we consider in this paper, could be thought of as a series of infinitesimal Dutch auctions of a fungible token, with new auctions being initiated at a linear rate over time, and each auction independently decaying in price at an exponential rate.

[1] One exception is that if other bidders attempt to submit a bid at around the same time as the winning bid, their transaction may be publicized and/or included on chain after the winning transaction, possibly paying fees.

Arbitrage profits. Dutch auctions have some drawbacks when implemented on a blockchain. In particular, since blocks only arrive at discrete times, the true market price of the asset at the time a block is created may be higher than the price offered by the auction, due to the decay of the Dutch auction price and the drift and volatility of the asset. This means the seller should expect to sell the asset at a discount to the market price or fair value at the time of sale, with the profits going to arbitrageurs or whoever is able to capture value from ordering transactions in the block — a type of maximal extractable value (MEV).

This type of loss is similar to the "quote-sniping" losses of market makers in high-frequency trading models [Budish et al., 2015], or the "loss-versus-rebalancing" suffered by liquidity providers on automated market makers, a concept introduced by Milionis et al. [2022]. In Milionis et al. [2023], LVR was extended to incorporate discrete blocks. For analytic tractability, block generation times are assumed to be from a Poisson process, an assumption we also make here.

Contributions. In this paper, we apply a similar model to Dutch auctions and gradual Dutch auctions. Given certain parameters for a Dutch auction — volatility, drift, starting price, decay rate, and average block arrival times — we derive closed-form expressions for both the losses to fair value and expected time-to-fill. We also extend the analysis to gradual Dutch auctions, showing how expected losses to arbitrageurs and expected sales rate vary as a function of these parameters.

For both Dutch auctions and gradual Dutch auctions, as long as the auction starts above the current price, LVF is given by the following expression (where δ is the decay rate of the auction plus the asset's drift in log space, σ is the volatility of the asset, and Δt is the mean interblock time):

$$\mathsf{LVF}_+ = \frac{1}{1 + \frac{\delta}{\sigma^2}\left(\sqrt{1 + \frac{2\sigma^2}{\delta^2 \Delta t}} - 1\right)}.$$

For example, if volatility is 5% per day (0.017% per second), decay rate is 0.01% per second, and average block time is 12 seconds, LVF_+ is about 0.13%. This would mean that for every \$100 worth of tokens that they sell, the seller should expect to get about \$99.87.

For gradual Dutch auctions, the rate at which tokens are sold is simply proportional to the drift δ. For regular Dutch auctions, the amount of time to fill if the starting price of the auction is higher than the current price is given by the following formula, in which z_0 is the (log) difference between the starting price and the current price:

$$\mathsf{FT}(z_0) = \frac{z_0}{\delta} + \frac{\Delta t}{2}\left(1 + \sqrt{1 + \frac{2\sigma^2}{\delta^2 \Delta t}}\right).$$

For example, with the same parameters as above, and with starting price 0.1% higher than the current price, the expected time to fill is about 23.3 seconds.

We also find closed forms for LVF and FT in the cases where the starting price of the auction is below the current price.

These models show how changing the decay rate of the auction affects both speed of execution and expected loss, helping inform practitioners who want to trade off between those values when choosing auction parameters such as initial price and decay rate. They also show how the characteristics of the blockchain — particularly average block time — affect the efficiency of Dutch auctions. For example, the formula for LVR_+ above satisfies the lower bound

$$\mathsf{LVF}_+ \geq \frac{1}{1 + \frac{1}{\sigma\sqrt{\Delta t/2}}} \approx \sigma\sqrt{\Delta t/2},$$

where the approximation holds for Δt small (the "fast block" regime). This suggests that if a platform wants to support Dutch auctions that lose less than 2 basis points for assets with daily volatility of 5%, it will need to have block times of less than 2.75 seconds.

1.1. Literature Review

Dutch auctions have been analyzed extensively in the auction theory and mechanism design literature, since at least the work of Vickrey [1961], who showed the strategic equivalence of Dutch auctions and first-price sealed-bid auctions under certain assumptions.

Our approach is related to barrier-diffusion approaches [Hasbrouck, 2007] to limit order pricing. For example, Lo et al. [2002] model the time-to-fill for a limit order as the first-passage time for a geometric Brownian motion with drift, and solve for the distribution of this time. Mathematically, this is equivalent to a continuous time version of our model.[2] Crucially, they do not consider loss-versus-fair for a limit order, since this quantity would be zero in continuous time. Moallemi and Sağlam [2013] consider frictions introduced by latency in submitting limit orders, also using a barrier-diffusion model. The central novelty of the present paper is the blockchain setting: we analyze frictions restricting the ability to trade in the auction introduced by the discrete block generation process. To our knowledge, no prior work has modeled the behavior of Dutch auctions for geometric Brownian motion assets with discrete block generation times.

The idea of gradual Dutch auctions was proposed by Frankie et al. [2022]. Transmissions11 et al. [2022] proposed an extension on the idea, variable rate GDAs, in which the target sales rate could vary as a function of time. Gradual Dutch auctions could be thought of as similar to automated market makers (AMMs) for which the price impact function is an exponential function, the fee to buy is 0, the fee to sell is infinite, and the asset price has a negative drift. In this way, we build on the setting of Milionis et al. [2023] in computing arbitrage profits for AMMs with fees.

A version of the GDA mechanism was studied by Kulkarni et al. [2023]. That work considers the use of discrete GDAs for illiquid NFTs where buyers depend on private signals for valuation, rather than continuous GDAs for highly liquid fungible tokens driven by common valuations.

[2] In our setting, the drift arises from descending price of the Dutch auction, while for Lo et al. [2002], the limit order is at a static price and the drift arises from the underlying asset price process.

2. Model

We imagine a scenario where an agent is selling[3] a risky asset via a descending price Dutch auction in a common value setting. Following the model of Milionis et al. [2023], we assume there exists the common fundamental value or price P_t at time t that follows a geometric Brownian motion price process,

$$\frac{dP_t}{P_t} = \mu\, dt + \sigma\, dW_t, \tag{1}$$

where $\{W_t\}$ is a Brownian motion, and the process is parameterized by drift μ and volatility $\sigma > 0$.

The agent is progressively willing to lower their offered price. Let A_t denote lowest price the agent is willing to sell at, at time t, i.e., the best ask price. We assume A_t decreases exponentially according to[4]

$$\frac{dA_t}{A_t} = -\lambda\, dt, \tag{2}$$

with decay constant $\lambda > 0$. Define the log mispricing process $z_t \triangleq \log(A_t/P_t)$, so that, applying Itô's lemma,

$$dz_t = -\underbrace{\left(\lambda + \mu - \tfrac{1}{2}\sigma^2\right)}_{\triangleq \delta} dt + \sigma\, dW_t.$$

We assume that blocks are generated according to a Poisson process[5] of rate Δt^{-1}, where $\Delta t > 0$ is the mean interblock time. We assume there is a population of "arbitrageurs", or traders informed about the fundamental price P_t, and will buy from the agent at any discount to this price. However, these agents can only act at block generation times. Thus, if τ is a block generation time, and $A_{\tau-} < P_{\tau-}$, arbs trade until there is no marginal profit, so that $A_\tau = P_\tau$ and $z_\tau = 0$ Thus, we have $z_\tau = \max(0, z_{\tau_-})$. Then, $\{z_t\}$ is a Markov jump diffusion process. Since it involves the interaction of a diffusive process $\{z_t\}$ with a barrier ($z_t = 0$), our model falls into the general class of barrier-diffision models for market microstructure [Hasbrouck, 2007].

We will make the following assumption:

Assumption 1. *Assume that* $\delta \triangleq \lambda + \mu - \frac{1}{2}\sigma^2 > 0$.

This assumption is sufficient to ensure that trade occurs with probability 1, and necessary so

[3]While we focus on the case of an agent selling the asset via a Dutch auction, our model also applies to the case of an agent buying via an ascending price Dutch auction-style mechanism. In that case, the mechanism would have a steadily increasing bid price at which it is willing to buy the asset, and analogous formulas could be obtained. Note that over longer time horizons, the two cases are not completely symmetric because of the positivity of prices and the inherent asymmetry of geometric Brownian model. In particular, for example, for a seller LVF is bounded above by 100% since the sale price will always be bounded below by zero, while LVF is unbounded above for a buyer, since the buy price is unbounded above.

[4]We choose exponentially decreasing prices because it matches well with the geometric Brownian motion price dynamics (1). An alternative choice would be to assume the ask price decreases linearly and that the price process is a arithmetic Brownian motion. On the short timescales of practical interest, this would yield similar results both quantitatively and qualitatively to the model here.

[5]For a proof-of-work blockchain, Poisson block generation is a natural assumption [Nakamoto, 2008]. However, modern proof-of-state blockchains typically generate blocks at deterministic times. In these cases, we will view the Poisson assumption as an approximation that is necessary for tractability.

that that the expected time to trade is finite. It can be satisfied by the agent making a sufficiently large choice of the decay rate λ. Under Assumption 1, the following lemma gives the stationary distribution $\pi(z)$ of z_t:[6]

Lemma 1. *If $\delta > 0$, the process z_t is an ergodic process on $\mathbb{R}$, with unique invariant distribution $\pi(\cdot)$ given by the density*

$$p_\pi(z) = \begin{cases} \pi_+ \times p^{\text{exp}}_{\zeta_+}(z) & \text{if } z \geq 0, \\ \pi_- \times p^{\text{exp}}_{\zeta_-}(-z) & \text{if } z < 0, \end{cases}$$

for $z \in \mathbb{R}$. Here, $p^{\text{exp}}_\zeta(z) \triangleq \zeta e^{-\zeta z}$ is the density of an exponential distribution over $z \in \mathbb{R}_+$ with parameter $\zeta > 0$. The parameters $\{\zeta_\pm\}$ are given by

$$\zeta_- \triangleq \frac{\delta}{\sigma^2}\left(\sqrt{1 + \frac{2\sigma^2}{\delta^2 \Delta t}} - 1\right), \quad \zeta_+ \triangleq \frac{2\delta}{\sigma^2}.$$

The probabilities $\{\pi_\pm\}$ are given by

$$\pi_- \triangleq \pi\big((-\infty, 0)\big) = \delta \Delta t \zeta_-, \quad \pi_+ \triangleq \pi\big([0, +\infty)\big) = 1 - \delta \Delta t \zeta_-.$$

3. Regular Dutch Auctions

We first consider the case of a discrete quantity of the risk asset for sale, initially at ask price A_0, or, alternatively, initial log mispricing $z_0 \triangleq \log(A_0/P_0)$, with the ask price A_t decreasing exponentially at rate λ according to (2). Suppose the order is traded at fill time τ_F, i.e., τ_F is the earliest block generation time which satisfies $z_{\tau_F} \leq 0$. Then, the order will sell at price A_{τ_F} when the fundamental value is P_{τ_F}. We are interested in the expected relative loss versus the fundamental price or fair value, i.e.,

$$\frac{P_{\tau_F} - A_{\tau_F}}{P_{\tau_F}} = 1 - e^{z_{\tau_F}}.$$

Loss-versus-fair and time-to-fill. We are interested in the expected relative loss, which we call "loss-versus-fair" (LVF), i.e.,

$$\mathsf{LVF}(z_0) \triangleq \mathsf{E}\left[1 - e^{z_{\tau_F}} \,|\, z_0\right].$$

We are also interested in the expected time-to-fill, i.e.,

$$\mathsf{FT}(z_0) \triangleq \mathsf{E}\left[\tau_F | \, z_0\right].$$

The following theorem characterizes these quantities:

[6]While applied in a different context, Lemma 1 is a special case of Theorem 7 of Milionis et al. [2023] up to a sign change, with $\gamma_- \to \infty$. For completeness, a standalone proof is provided in Appendix A.

Theorem 1. *If $z_0 \geq 0$, the expected relative loss and time-to-fill are given by*

$$\mathsf{LVF}(z_0) = \frac{1}{1+\zeta_-} = \frac{1}{1 + \frac{\delta}{\sigma^2}\left(\sqrt{1+\frac{2\sigma^2}{\delta^2 \Delta t}} - 1\right)} \triangleq \mathsf{LVF}_+, \tag{3}$$

$$\mathsf{FT}(z_0) = \frac{z_0}{\delta} + \frac{\Delta t}{2}\left(1 + \sqrt{1 + \frac{2\sigma^2}{\delta^2 \Delta t}}\right). \tag{4}$$

If $z_0 < 0$, then

$$\mathsf{LVF}(z_0) = 1 - \frac{e^{z_0}}{1 + \Delta t\left(\delta - \frac{1}{2}\sigma^2\right)} + \left(\frac{1}{1+\zeta_-} - \frac{\Delta t\left(\delta - \frac{1}{2}\sigma^2\right)}{1 + \Delta t\left(\delta - \frac{1}{2}\sigma^2\right)}\right) e^{\zeta_- z_0}, \tag{5}$$

$$\mathsf{FT}(z_0) = \frac{\Delta t}{2}\left(2 + \left(\sqrt{1 + \frac{2\sigma^2}{\delta^2 \Delta t}} - 1\right) e^{\zeta_- z_0}\right). \tag{6}$$

The formulas of Theorem 1 are illustrated for representative parameter choices in Figure 1.

Discussion of loss-versus-fair. Observe that, for $z_0 \geq 0$, the loss is given by $\mathsf{LVF}(z_0) = \mathsf{LVF}_+$ and does not depend on the initial mispricing z_0. This is because, starting at $z_0 \geq 0$, the mispricing process must first pass through the boundary $z_t = 0$, since it is continuous. If we denote by τ_0 the first passage time of that boundary, because $\{z_t\}$ is a Markov process and block generation times are memoryless, we have that $\mathsf{LVF}(z_0) = \mathsf{E}[\mathsf{LVF}(z_{\tau_0})] = \mathsf{LVF}(0) = \mathsf{LVF}_+$. For $z_0 < 0$, $\mathsf{LVF}(z_0)$ is strictly decreasing in z_0. This is intuitive: the more the asset is initially underpriced, the larger the expected losses experienced upon the eventual sale.

Now, consider properties of the loss LVF_+. Observe that this is a strictly increasing function of the mispricing δ, so that it is minimized when $\delta = 0$, and we have the lower bound

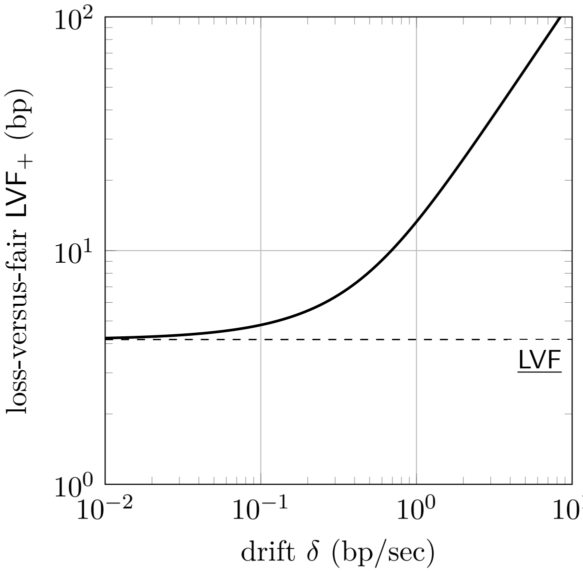
$$\underline{\mathsf{LVF}} \triangleq \frac{1}{1 + \frac{1}{\sigma\sqrt{\Delta t/2}}} \leq \mathsf{LVF}_+ \leq \mathsf{LVF}(z_0). \tag{7}$$

In general, setting as small a value of the drift δ as possible minimizes losses. However, the left side of (7) yields a lower bound on the loss that is due intrinsic volatility and discrete blocks. Indeed, in the fast block regime, when the average interblock time Δt is small, this lower bound takes the form

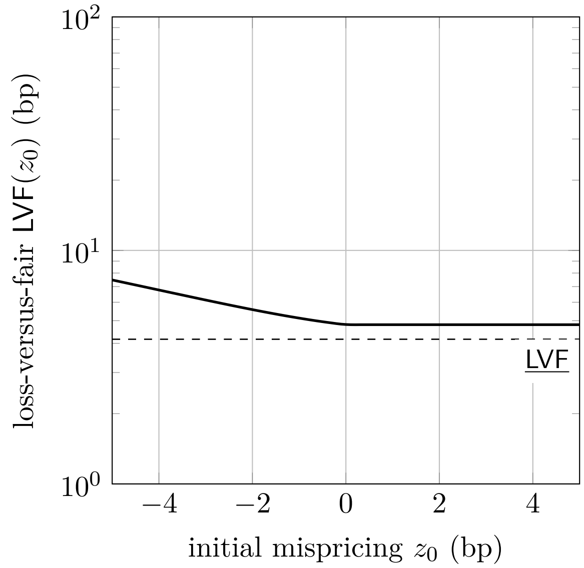
$$\underline{\mathsf{LVF}} \triangleq \frac{1}{1 + \frac{1}{\sigma\sqrt{\Delta t/2}}} \approx \sigma\sqrt{\Delta t/2},$$

which is the standard deviation of changes in the mispricing process over half of a typical interblock time. This is a minimum, unavoidable level of loss, no matter what choice of auction parameters (z_0, δ) is made.

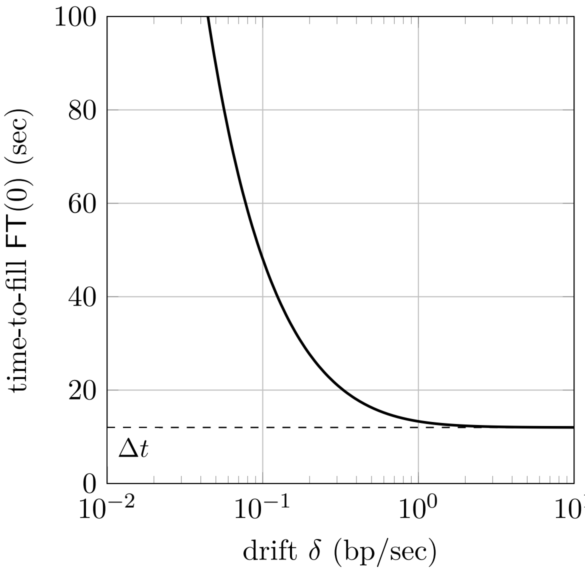
Discussion of time-to-fill. For the time-to-fill, observe that $\mathsf{FT}(z_0)$ is a strictly increasing function



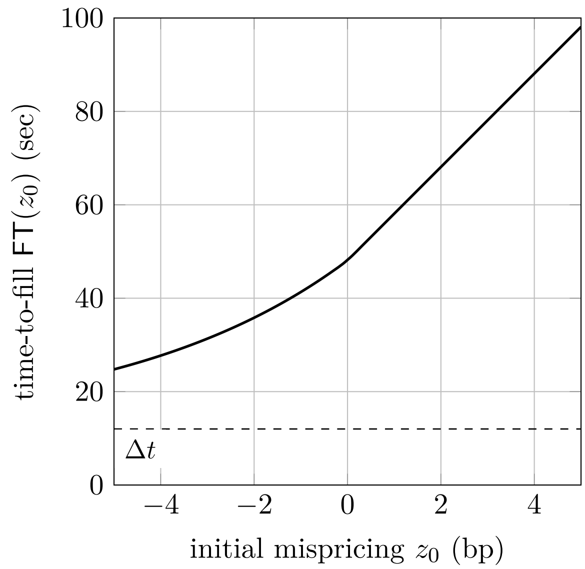
(a) $\mathsf{LVF}_+ = \mathsf{LVF}(0)$ as a function of δ.



(b) $\mathsf{LVF}(z_0)$ as a function of z_0, assuming a fixed value of $\delta = 0.1$ (bp/sec).



(c) $\mathsf{FT}(0)$ as a function of δ.



(d) $\mathsf{FT}(z_0)$ as a function of z_0, assuming a fixed value of $\delta = 0.1$ (bp/sec).

Figure 1: Comparison of LVF and FT for different parameter choices. These figures assume $\sigma = 5\%$ (daily) and $\Delta t = 12$ (sec). The dashed lines correspond to the lower bounds of (7) and (8).

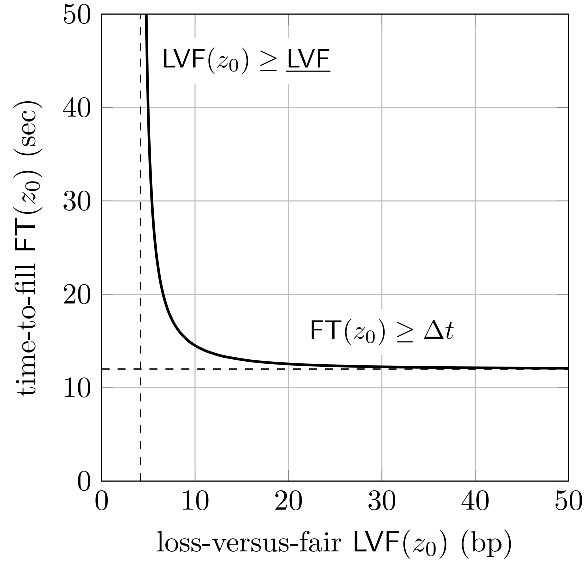


Figure 2: The efficient frontier trading off loss-versus-fair and time-to-fill. This figure assumes $\sigma = 5\%$ (daily) and $\Delta t = 12$ (sec). The dashed lines correspond to the lower bounds of (7) and (8).

of the initial mispricing z_0 and a strictly decreasing function of the drift δ, and that

$$\mathsf{FT}(z_0) \geq \lim_{z \to -\infty} \mathsf{FT}(z) = \Delta t. \tag{8}$$

This lower bound is intuitive: by the memoryless nature of the Poisson process, the time-to-fill is always lower bounded by the mean interblock time.

Parameter optimization (known value). Theorem 1 can be applied to optimize the initial auction price A_0 at time $t = 0$ and the decay rate λ. When the initial value P_0 is known, we will parameterize this decision with the variables $z_0 \triangleq \log(A_0/P_0)$ and $\delta \triangleq \lambda + \mu - \frac{1}{2}\sigma^2 > 0$. Then, the seller can solve the optimization problem

$$\underset{z_0, \delta \geq 0}{\text{minimize}} \ \mathsf{LVF}(z_0) + \theta \cdot \mathsf{FT}(z_0).$$

Here, $\theta \geq 0$ is a parameter that captures the trade off between minimizing loss and time-to-fill. The efficient frontier of Pareto optimal outcomes with these two objectives can be generated by varying θ. An example of such an efficient frontier is illustrated in Figure 2.

Note that, in this setting, is never optimal to pick $z_0 > 0$. This is because such a choice of z_0 is Pareto dominated by setting $z_0 = 0$: in this case lowering the value of z_0 strictly decreases $\mathsf{FT}(z_0)$, without increasing $\mathsf{LVF}(z_0)$. Indeed, with the representative parameter choices of Figure 2, setting $z_0 \approx 0$ is typically optimal, i.e., the auction should be started at the current fundamental value (when it is known).

Parameter optimization (unknown value). Another setting of interest is where the buyer is uncertain of the value P_0 when determining the auction parameters. We describe this uncertainty with a lognormal Bayesian prior: assume the seller believes that $P_0 \sim \hat{P}_0 e^{-\frac{1}{2}\sigma_0^2 + \sigma_0 Z}$, where $Z \sim N(0,1)$, $\hat{P}_0 = \mathsf{E}[P_0]$ is the mean of the prior belief, and $\sigma_0 > 0$ is the volatility of the prior belief. Then, we have $z_0 = \log A_0/\hat{P}_0 + \frac{1}{2}\sigma_0^2 - \sigma_0 Z$. Then, the seller can compute the loss-versus-fair and time-to-fill efficient frontier by solving the optimization problem

$$\underset{A_0, \delta \geq 0}{\text{minimize}} \;\; \mathsf{E}\left[\mathsf{LVF}\left(\log A_0/\hat{P}_0 + \tfrac{1}{2}\sigma_0^2 - \sigma_0 Z\right)\right] + \theta \cdot \mathsf{E}\left[\mathsf{FT}\left(\log A_0/\hat{P}_0 + \tfrac{1}{2}\sigma_0^2 - \sigma_0 Z\right)\right],$$

for varying values of $\theta \geq 0$. Note that the expectations in the objective function can be computed in closed form, these formulas are provided in Appendix B.

4. Gradual Dutch Auctions

In this section, we develop stationary, steady-state analogs of the results of Section 3 in the context of gradual Dutch auctions. Introduced by Frankie et al. [2022], the continuous gradual Dutch auctions we consider here continously emit the risky asset for sale at a rate per unit time given by $r > 0$. Each emission is in turn are sold through a Dutch auction where the price decreases exponentially with decay rate $\lambda > 0$. Our goal will be to compute the steady-state rate at which such auctions leak value to arbitrageurs, as well as the rate of trade. We will see a similar tradeoff as in Section 3

In our stationary, steady-state setting, we will imagine that the seller has been continuously emitting auctions since time $t = -\infty$. At any time t, if an auction has age u, the auction price is given by $ke^{-\lambda u}$, for some constant $k > 0$. When the the age of the oldest available auction is T, this auction defines the best ask price by $A_t = ke^{-\lambda T}$. Hence, if an agent wishes to purchase a total quantity q at time t, and the age of the oldest available auction is T, the total cost is given by

$$C_t(q) = \int_{T-q/r}^{T} ke^{-\lambda u} \cdot r\, dt = \frac{kr}{\lambda} \frac{e^{\lambda q/r} - 1}{e^{\lambda T}} = A_t \cdot \frac{r}{\lambda} \left(e^{\lambda q/r} - 1\right).$$

Denote the block generation times by $0 < \tau_1 < \tau_2 < \cdots$. When a block is generated at each time $t = \tau_i$, arbitraguers can trade against the auctions, and will myopically seek to do so to maximize their instananeous profit, assuming they value the risky asset at the current fundamental price P_t. The following lemma characterizes this behavior:

Lemma 2. *Suppose a block is generated at time τ, with current fundamental price given by $P \triangleq P_\tau$, and mispricing (immediately before block generation) given by $z \triangleq z_{\tau-}$. Then, if $\lambda > 0$, the optimal arbitrage trade quantity of the risky asset is given by*

$$q^*(z) \triangleq -\frac{r}{\lambda} z \mathbb{I}_{\{z \leq 0\}},$$

with optimal arbitrage profits (or, equivalently, the total loss experienced by the auction seller rela-

tive to selling at the current fair fundamental price P)

$$A^*(P, z) \triangleq \frac{Pr}{\lambda} \{e^z - 1 - z\} \mathbb{I}_{\{z \le 0\}}.$$

Proof. The arbitrageur faces the maximization problem

$$\underset{q \ge 0}{\text{maximize}} \quad P_\tau q - C_{\tau -}(q) = P \left\{ q - \frac{e^z r}{\lambda} \left(e^{\lambda q / r} - 1 \right) \right\},$$

where we use the fact that $A_{\tau -} = P_\tau e^{z_{\tau -}}$. The result follows from straightforward analysis of the first order and second order conditions for this optimization problem. Note that that $\lambda > 0$ is required for the second order conditions (concavity). ■

Denote by N_T the total number of block generated over the time interval $[0, T]$. Suppose an arbitrageur arrives at time τ_i, observing external price P_{τ_i} and mispricing $z_{\tau_i^-}$. From Lemma 2, the arbitrageur profit is given by $A^*(P_{\tau_i}, z_{\tau_i^-})$ and the trade size is given by $q^*(z_{\tau_i^-})$. We can write the total arbitrage profit and total quantity traded (measured in the numéraire) paid over $[0, T]$ by summing over all arbitrageurs arriving in that interval, i.e.,

$$\mathsf{ARB}_T \triangleq \sum_{i=1}^{N_T} A^*(P_{\tau_i}, z_{\tau_i^-}), \quad \mathsf{VOL}_T \triangleq \sum_{i=1}^{N_T} P_{\tau_i} q^*(z_{\tau_i^-}).$$

Clearly these are non-negative and monotonically increasing jump processes. The following theorem characterizes their instantaneous expected rate of growth or intensity:[7]

Theorem 2 (Rate of Arbitrage Profit and Volume). *Define the intensity, or instantaneous rate of arbitrage profit and volume, by*

$$\overline{\mathsf{ARB}} \triangleq \lim_{T \to 0} \frac{\mathsf{E}[\mathsf{ARB}_T]}{T}, \quad \overline{\mathsf{VOL}} \triangleq \lim_{T \to 0} \frac{\mathsf{E}[\mathsf{VOL}_T]}{T}.$$

Given initial price $P_0 = P$, suppose that $z_{0-} = z$ is distributed according to its stationary distribution $\pi(\cdot)$. Then, the instantaneous rate of arbitrage profit and volume are given by

$$\overline{\mathsf{ARB}} = \frac{\mathsf{E}_\pi [A^*(P, z)]}{\Delta t} = \frac{Pr\delta}{\delta - \mu + \frac{1}{2}\sigma^2} \times \frac{1}{1 + \zeta_-}, \tag{9}$$

$$\overline{\mathsf{VOL}} = \frac{\mathsf{E}_\pi [Pq^*(z)]}{\Delta t} = \frac{Pr\delta}{\delta - \mu + \frac{1}{2}\sigma^2}. \tag{10}$$

Comparing the instantaneous rate of arbitrage profit and volume given by (9)–(10) with Theorem 1, we have that

$$\overline{\mathsf{ARB}} = \overline{\mathsf{VOL}} \times \mathsf{LVF}_+. \tag{11}$$

[7] Mathematically, $\overline{\mathsf{ARB}}$ is the intensity of the compensator for the monotonically increasing jump process ARB_T at time $T = 0$, similarly $\overline{\mathsf{VOL}}$ is the intensity of the compensator for VOL_T.

This expression highlights the fact a gradual Dutch auction can be viewed as a continuum of many regular Dutch auctions, each of infinitesimal size, and each at a different price. From Theorem 1, we know that the seller will incur the same expected relative loss per dollar sold, LVF_+, in each of these auctions. This loss is the same irrespective of the different prices because all of the auctions start out-of-the-money ($z_0 \geq 0$). Equation (11) intuitively decomposes the total arb profits per unit time as the product of the dollar volume sold per unit time and the loss per dollar sold.

Parameter optimization. As in Section 3, we can leverage Theorem 2 to optimize parameter choice in a gradual Dutch auction. In particular, a gradual Dutch auction is parameterized by the choice of emission rate $r \geq 0$ and the choice of drift $\delta \triangleq \lambda + \mu - \frac{1}{2}\sigma^2$ satisfying $\delta \geq 0$ and $\lambda = \delta - \mu + \frac{1}{2}\sigma^2 > 0$ (the second condition is required for concavity in Lemma 2). This choice can be made to minimize the losses incurred while maximizing the rate of trade. For example, consider the optimization problem

$$\underset{r>0,\ \delta\geq\max(0,\mu-\frac{1}{2}\sigma^2)}{\text{minimize}} \quad \mathsf{LVF}_+ - \theta \cdot \overline{\mathsf{VOL}},$$

where $\theta \geq 0$ is a tradeoff parameter.

5. Conclusion and Future Work

While there has been an increasing amount of academic interest in studying, designing, and formalizing automated market makers for liquid assets in the blockchain context, there has been somewhat less attention paid to Dutch auctions, despite their popularity with protocol implementers. This paper was an attempt to bring the theoretical understanding of Dutch auctions in the setting of discrete block generation times closer to the current level of understanding that has been reached around automated market makers, particularly in Milionis et al. [2022] and Milionis et al. [2023].

The paper also sought to provide a guide for application designers in setting parameters for Dutch auctions, including deriving formulas that map the tradeoff between speed of execution and quality of execution. The paper may also be helpful for platform designers in determining performance parameters like block times. For example, the rule of thumb in Equation (7) suggests that if a platform wants to support Dutch auctions that lose less than 2 basis points for assets with daily volatility of 5%, it will need to have block times of less than 2.75 seconds.

The model in this paper shared some of the limitations of the model in Milionis et al. [2023], including not taking into account fixed transaction fees such as “gas” and use of a Poisson model for block generation as opposed to deterministic block generation, which is more relevant for modern proof-of-stake blockchains. Further, a purely diffusive, continuous process (geometric Brownian motion) has been used to model innovations in the fundamental price process, while jumps are are known to be an important component of high-frequency price dynamics. Additionally, while this work quantified the losses inherent in Dutch auctions, it does not explore possible alternative designs for Dutch auctions that might mitigate those losses without reducing the speed of execution. We hope further work can explore this area.

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A. Proofs

Proof of Lemma 1. Note that $\{z_t\}$ is a Markov jump diffusion process, with infinitesimal generator

$$\mathcal{A}f(z) = \tfrac{1}{2}\sigma^2 f''(z) - \delta f'(z) + \Delta t^{-1}\left[f(0) - f(z)\right]\mathbb{I}_{\{z<0\}},$$

given a test function $f\colon \mathbb{R} \to \mathbb{R}$.

The invariant distribution $\pi(\cdot)$ must satisfy

$$\mathsf{E}_\pi[\mathcal{A}f(z)] = \int_{-\infty}^{+\infty} \mathcal{A}f(z)\,\pi(dz) = 0, \tag{12}$$

for all test functions $f\colon \mathbb{R} \to \mathbb{R}$. We will guess that $\pi(\cdot)$ decomposes according to two different densities over the positive and negative half line, and then compute the conditional density on each segment via Laplace transforms using (12).

Consider the test function

$$f_-(z) \triangleq \begin{cases} e^{\alpha z} & \text{if } z < 0, \\ 1 + \alpha z & \text{if } z \geq 0. \end{cases}$$

Then,

$$\mathcal{A}f_-(z) = \tfrac{1}{2}\sigma^2\alpha^2 e^{\alpha z}\mathbb{I}_{\{z<0\}} - \delta\alpha\left(e^{\alpha z}\mathbb{I}_{\{z<0\}} + \mathbb{I}_{\{z\geq 0\}}\right) + \Delta t^{-1}\left[1 - e^{\alpha z}\right]\mathbb{I}_{\{z<0\}},$$

so that

$$\begin{aligned} 0 &= \mathsf{E}_\pi\left[\mathcal{A}f_-(z)\right] \\ &= \tfrac{1}{2}\sigma^2\alpha^2\pi_-\mathsf{E}_\pi\left[e^{\alpha z}\middle|\, z<0\right] - \delta\alpha\left(\pi_-\mathsf{E}_\pi\left[e^{\alpha z}\middle|\, z<0\right] + \pi_+\right) + \Delta t^{-1}\pi_-\left(1 - \mathsf{E}_\pi\left[e^{\alpha z}\middle|\, z<0\right]\right). \end{aligned}$$

Then,

$$\mathsf{E}_\pi\left[e^{\alpha z}\middle|\, z<0\right] = \frac{\delta\alpha\frac{\pi_+}{\pi_-} - \Delta t^{-1}}{\frac{1}{2}\sigma^2\alpha^2 - \delta\alpha - \Delta t^{-1}}.$$

Observe the denominator has a single negative root. Then, $\pi(-z|z<0)$ must be exponential with parameter $\zeta_- \triangleq \left(\sqrt{\delta^2 + 2\Delta t^{-1}\sigma^2} - \delta\right)/\sigma^2$. Also, note that

$$\mathsf{E}_\pi[-z|z<0] = 1/\zeta_-.$$

Next consider the test function

$$f_+(z) \triangleq \begin{cases} e^{-\alpha z} & \text{if } z \geq 0, \\ 1 - \alpha z & \text{if } z < 0. \end{cases}$$

Then,

$$\mathcal{A}f_+(z) = \tfrac{1}{2}\sigma^2\alpha^2 e^{-\alpha z}\mathbb{I}_{\{z\geq 0\}} + \delta\alpha\left(e^{-\alpha z}\mathbb{I}_{\{z\geq 0\}} + \mathbb{I}_{\{z<0\}}\right) + \Delta t^{-1}\alpha z\mathbb{I}_{\{z<0\}},$$

so that

$$\begin{aligned}0 &= \mathsf{E}_\pi\left[\mathcal{A}f_+(z)\right] \\ &= \tfrac{1}{2}\sigma^2\alpha^2\pi_+\mathsf{E}_\pi\left[e^{-\alpha z}\middle|\, z\geq 0\right] + \delta\alpha\left(\pi_+\mathsf{E}_\pi\left[e^{-\alpha z}\middle|\, z\geq 0\right] + \pi_-\right) + \Delta t^{-1}\alpha\pi_-\mathsf{E}_\pi\left[z\middle|\, z<0\right].\end{aligned}$$

Then,

$$\begin{aligned}\mathsf{E}_\pi\left[e^{-\alpha z}\middle|\, z\geq 0\right] &= -\frac{\pi_-}{\pi_+}\frac{\delta + \Delta t^{-1}\mathsf{E}_\pi\left[z\middle|\, z<0\right]}{\frac{1}{2}\sigma^2\alpha+\delta} \\ &= -\frac{\pi_-}{\pi_+}\frac{\delta - \Delta t^{-1}/\zeta_-}{\frac{1}{2}\sigma^2\alpha+\delta}\end{aligned}$$

Then, $\pi(z|z\geq 0)$ must be exponential with parameter $\zeta_+ \triangleq 2\delta/\sigma^2$. Substituting $\alpha = 0$, we have

$$1 = -\frac{\pi_-}{\pi_+}\frac{\delta - \Delta t^{-1}/\zeta_-}{\delta} = -\frac{\pi_-}{1-\pi_-}\left(1 - \frac{1}{\delta\Delta t\zeta_-}\right).$$

Solving for π_-,

$$\pi_- = \delta\Delta t\zeta_-, \quad \pi_+ = 1 - \delta\Delta t\zeta_-.$$

■

Proof of Theorem 1. We consider $\mathsf{LVF}(z_0)$ and $\mathsf{FT}(z_0)$ separately.

Loss-versus-fair. We begin with the LVF calculation. First, consider the case where $z_0 \geq 0$. Define the τ_F to be the fill time of the order, i.e., the first Poisson block generation time τ_F with $z_{\tau_F} \leq 0$. Also define $\tau_0 \triangleq \min\{t\geq 0\colon z_t = 0\}$ to be first passage time for the boundary $z_t = 0$. Since the the process the mispricing process is continuous, we must have that $\tau_F \geq \tau_0$. Then,

$$\begin{aligned}\mathsf{LVF}(z_0) &\triangleq \mathsf{E}\left[1 - e^{z_{\tau_F}}\,\middle|\, z_0\right] \\ &\overset{\text{(a)}}{=} \mathsf{E}\left[\mathsf{E}\left[1 - e^{z_{\tau_F}}\,\middle|\, \tau_0, z_{\tau_0}\right]\middle|\, z_0\right] \\ &\overset{\text{(b)}}{=} \mathsf{E}\left[\mathsf{LVF}(z_{\tau_0})\middle|\, z_0\right] \\ &\overset{\text{(c)}}{=} \mathsf{LVF}(0) \triangleq \mathsf{LVF}_+.\end{aligned} \tag{13}$$

where (a) follows from the tower property of expectation, (b) follows from the fact that Poisson arrivals are memoryless and $\{z_t\}$ is a Markov process, and (c) follows from the fact that $z_{\tau_0} = 0$.

Now, consider arbitrary $z_0 \in \mathbb{R}$. Let $\tau_B > 0$ be the first Poisson block generation time. Since

$\tau_F \geq \tau_B$,

$$
\begin{aligned}
\mathsf{LVF}(z_0) &\triangleq \mathsf{E}\left[1 - e^{z_{\tau_F}} \,|\, z_0\right] \\
&\overset{\text{(a)}}{=} \mathsf{E}\left[\mathsf{E}\left[1 - e^{z_{\tau_F}} \,|\, \tau_B, z_{\tau_B}\right] \middle|\, z_0\right] \\
&\overset{\text{(b)}}{=} \mathsf{E}\left[\mathsf{LVF}(z_{\tau_B}) \middle|\, z_0\right] \\
&\overset{\text{(c)}}{=} \mathsf{E}\left[\mathsf{LVF}_+ \mathbb{I}_{\{z_{\tau_B} \geq 0\}} + (1 - e^{z_{\tau_B}})\, \mathbb{I}_{\{z_{\tau_B} < 0\}} \middle|\, z_0\right] \\
&\overset{\text{(d)}}{=} \int_0^\infty \frac{e^{-\tau/\Delta t}}{\Delta t} \int_0^{+\infty} \frac{1}{\sigma\sqrt{2\pi\tau}} e^{-\frac{1}{2}\left(\frac{z+\delta\tau - z_0}{\sigma\sqrt{\tau}}\right)^2} \mathsf{LVF}_+ \, dz \, d\tau \\
&\quad + \int_0^\infty \frac{e^{-\tau/\Delta t}}{\Delta t} \int_{-\infty}^{0} \frac{1}{\sigma\sqrt{2\pi\tau}} e^{-\frac{1}{2}\left(\frac{z+\delta\tau - z_0}{\sigma\sqrt{\tau}}\right)^2} (1 - e^z) \, dz \, d\tau.
\end{aligned}
\tag{14}
$$

where (a) follows from the tower property of expectation, (b) follows from the fact that Poisson arrivals are memoryless and $\{z_t\}$ is a Markov process, (c) follows from the fact that $\mathsf{LVF}(z_{\tau_B}) = \mathsf{LVF}_+$ for $z_{\tau_B} \geq 0$ while $\mathsf{LVF}(z_{\tau_B}) = 1 - e^{z_{\tau_B}}$ if $z_{\tau_B} < 0$, (d) follows from the fact that τ_B is exponentially distributed while, conditional on τ_B, z_{τ_B} is normally distributed, and $\Phi(\cdot)$ is the cumulative normal distribution. Substituting in $z_0 = 0$ and solving for $\mathsf{LVF}(z_0) = \mathsf{LVF}_+$, after integration, we obtain (3). For $z_0 \leq 0$, we can substitute (3) into (14) and integrate to obtain (5).

Time-to-fill. Suppose we start out at $z_0 \geq 0$, and define $\mathsf{FT}(z_0)$ to be the expected fill time of the next trade, i.e., the first Poisson arrival time τ with $z_{\tau_F} \leq 0$. Also define $\tau_0 = \min\{t \geq 0 \colon z_t = 0\}$ to be the first passage time for the boundary $z_t = 0$. Then, since the mispricing process $\{z_t\}$ is continuous and Markov, and Poisson arrivals are memoryless, we have that $\tau_F \geq \tau_0$ and

$$
\mathsf{FT}(z_0) = \mathsf{E}\left[\tau_F | z_0\right] = \mathsf{E}\left[\tau_0 | z_0\right] + \mathsf{E}\left[\mathsf{E}\left[\tau_F - \tau_0 |\, \tau_0, z_{\tau_0}\right] \middle|\, z_0\right] = \frac{z_0}{\delta} + \mathsf{FT}(0),
$$

where we have used the standard formula for expected first passage time of a Brownian motion with drift.

Thus, we have reduced to the case where $z_0 = 0$. Define $\tau_B > 0$ to be first Poisson block generation time. If $z_{\tau_B} \leq 0$, then τ_B is also the fill time. On the other hand, if $z_{\tau_B} > 0$, we will have to wait an additional amount after τ_B given in expectation by $\mathsf{FT}(z_{\tau_B}) = \mathsf{FT}(0)$. Thus, integrating over τ_B, and using the fact that, given τ_B, z_{τ_B} is normally distributed,

$$\begin{aligned}
\mathsf{FT}(0) &= \int_0^\infty \frac{e^{-\tau/\Delta t}}{\Delta t} \left(\tau + \int_0^\infty \frac{1}{\sigma\sqrt{2\pi\tau}} e^{-\frac{1}{2}\left(\frac{z+\delta\tau}{\sigma\sqrt{\tau}}\right)^2} \mathsf{FT}(z)\, dz \right) d\tau \\
&= \Delta t + \int_0^\infty \frac{e^{-\tau/\Delta t}}{\Delta t} \int_0^\infty \frac{1}{\sigma\sqrt{2\pi\tau}} e^{-\frac{1}{2}\left(\frac{z+\delta\tau}{\sigma\sqrt{\tau}}\right)^2} \left(\frac{z}{\delta} + \mathsf{FT}(0) \right) dz\, d\tau \\
&= \Delta t + \int_0^\infty \frac{e^{-\tau/\Delta t}}{\Delta t} \left\{ \int_0^\infty \frac{1}{\sigma\sqrt{2\pi\tau}} e^{-\frac{1}{2}\left(\frac{z+\delta\tau}{\sigma\sqrt{\tau}}\right)^2} \frac{z}{\delta}\, dz + \mathsf{FT}(0) \left(1 - \Phi\left(\frac{\delta\sqrt{\tau}}{\sigma} \right) \right) \right\} d\tau \\
&= \Delta t + \int_0^\infty \frac{e^{-\tau/\Delta t}}{\Delta t} \left\{ \int_0^\infty \frac{1}{\sigma\sqrt{2\pi\tau}} e^{-\frac{1}{2}\left(\frac{z+\delta\tau}{\sigma\sqrt{\tau}}\right)^2} \frac{z}{\delta}\, dz + \mathsf{FT}(0)\, \Phi\left(-\frac{\delta\sqrt{\tau}}{\sigma} \right) \right\} d\tau.
\end{aligned}$$

We can solve this for $\mathsf{FT}(0)$, i.e.,

$$\mathsf{FT}(0) = \frac{\Delta t + \int_0^\infty \frac{e^{-\tau/\Delta t}}{\Delta t} \int_0^\infty \frac{1}{\sigma\sqrt{2\pi\tau}} e^{-\frac{1}{2}\left(\frac{z+\delta\tau}{\sigma\sqrt{\tau}}\right)^2} \frac{z}{\delta}\, dz\, d\tau}{1 - \int_0^\infty \frac{e^{-\tau/\Delta t}}{\Delta t} \Phi\left(-\frac{\delta\sqrt{\tau}}{\sigma} \right) d\tau} = \frac{\Delta t}{2} \left(1 + \sqrt{1 + \frac{2\sigma^2}{\delta^2 \Delta t}} \right),$$

where the final equality is obtained via integration. This establishes (4).

Finally, consider the case where $z_0 < 0$. Fefine $\tau_B > 0$ to be first Poisson block generation time. If $z_{\tau_B} \leq 0$, then τ_B is also the fill time. On the other hand, if $z_{\tau_B} > 0$, we will have to wait an additional amount after τ_B given in expectation by $\mathsf{FT}(z_{\tau_B})$. Thus,

$$\begin{aligned}
\mathsf{FT}(z_0) &= \int_0^\infty \frac{e^{-\tau/\Delta t}}{\Delta t} \left(\tau + \int_0^\infty \frac{1}{\sigma\sqrt{2\pi\tau}} e^{-\frac{1}{2}\left(\frac{z+\delta\tau-z_0}{\sigma\sqrt{\tau}}\right)^2} \mathsf{FT}(z)\, dz \right) d\tau \\
&= \Delta t + \int_0^\infty \frac{e^{-\tau/\Delta t}}{\Delta t} \int_0^\infty \frac{1}{\sigma\sqrt{2\pi\tau}} e^{-\frac{1}{2}\left(\frac{z+\delta\tau-z_0}{\sigma\sqrt{\tau}}\right)^2} \left(\frac{z}{\delta} + \mathsf{FT}(0) \right) dz\, d\tau \\
&= \Delta t + \int_0^\infty \frac{e^{-\tau/\Delta t}}{\Delta t} \left\{ \int_0^\infty \frac{1}{\sigma\sqrt{2\pi\tau}} e^{-\frac{1}{2}\left(\frac{z+\delta\tau-z_0}{\sigma\sqrt{\tau}}\right)^2} \frac{z}{\delta}\, dz + \mathsf{FT}(0)\, \Phi\left(-\frac{\delta\tau - z_0}{\sigma\sqrt{\tau}} \right) \right\} d\tau.
\end{aligned}$$

After integration, this yields (6). ■

Proof of Theorem 2. We follow the method of Milionis et al. [2023]. Specifically, using the smoothing formula, e.g., Theorem 13.5.7 of Brémaud [2020],

$$\mathsf{E}\left[\mathsf{ARB}_T\right] = \mathsf{E}\left[\sum_{i=1}^{N_T} A^*(P_{\tau_i}, z_{\tau_i -}) \right] = \mathsf{E}\left[\int_0^T A^*(P_t, z_{t-})\, dN_t \right] = \mathsf{E}\left[\int_0^T A^*(P_t, z_{t-}) \cdot \Delta t^{-1}\, dt \right].$$

Applying Tonelli's theorem and the fundamental theorem of calculus,

$$\overline{\mathsf{ARB}} \triangleq \lim_{T\to 0} \frac{\mathsf{E}\left[\mathsf{ARB}_T\right]}{T} = \lim_{T\to 0} \frac{1}{T}\int_0^T \mathsf{E}\left[\frac{A^*(P_t, z_{t-})}{\Delta t}\right] dt = \frac{\mathsf{E}\left[A^*(P_0, z_{0-})\right]}{\Delta t} = \frac{\mathsf{E}_\pi\left[A^*(P, z)\right]}{\Delta t},$$

where in the final expression, $P_0 = P$ and z is distributed according to the stationary distribution $\pi(\cdot)$.

Then, using the definition of $\pi(\cdot)$ from Lemma 1 and $A^*(\cdot,\cdot)$ from Lemma 2,

$$\begin{aligned}
\overline{\mathsf{ARB}} &= \frac{1}{\Delta t}\pi(z|z\leq 0)\frac{Pr}{\lambda}\mathsf{E}_\pi\left[e^z - 1 - z\right|z\leq 0] \\
&= \frac{Pr}{\lambda}\delta\zeta_-\left\{\frac{\zeta_-}{1+\zeta_-} - 1 + \frac{1}{\zeta_-}\right\} \\
&= \frac{Pr}{\lambda} \times \frac{\delta}{1+\zeta_-},
\end{aligned}$$

as desired.

The same argument establishes that

$$\overline{\mathsf{VOL}} \triangleq \lim_{T\to 0} \frac{\mathsf{E}\left[\mathsf{VOL}_T\right]}{T} = \frac{\mathsf{E}_\pi\left[Pq^*(z)\right]}{\Delta t}.$$

Then, using Lemma 1 and $q^*(\cdot)$ from Lemma 2,

$$\begin{aligned}
\overline{\mathsf{VOL}} &= \frac{1}{\Delta t}\pi(z|z\leq 0)\frac{Pr}{\lambda}\mathsf{E}_\pi\left[-z\right|z\leq 0] \\
&= \frac{Pr}{\lambda} \times \delta,
\end{aligned}$$

as desired. ■

B. Formulas Under Fundamental Value Uncertainty

Assume that that the prior belief on the initial mispricing is normally distributed, i.e., $z_0 \sim N(\mu_0, \sigma_0^2)$. Then, via direct integration,

$$\begin{aligned}
\mathsf{E}\left[\mathsf{LVF}\left(z_0\right)\right] = \mathsf{LVF}_+ &+ \left(1 - \mathsf{LVF}_+\right)\Phi\left(-\frac{\mu_0}{\sigma_0}\right) + \frac{\Delta t^{-1} e^{\frac{\sigma_0^2}{2}}}{\frac{\sigma^2}{2} - \delta - \Delta t^{-1}} e^{\mu_0}\Phi\left(-\sigma_0 - \frac{\mu_0}{\sigma_0}\right) \\
&+ \left\{\left(\mathsf{LVF}_+ - \frac{\frac{\sigma^2}{2} - \delta}{\frac{\sigma^2}{2} - \delta - \Delta t^{-1}}\right) e^{\frac{\Delta t^{-1}\sigma_0^2}{\sigma^2} + \frac{\delta}{\sigma^2}\left(\frac{\delta}{\sigma^2}\sigma_0^2 + \mu_0\right)\left(\sqrt{1 + \frac{2\Delta t^{-1}\sigma^2}{\delta^2}} + 1\right)} \right. \\
&\left. \times\, \Phi\left(-\frac{\delta\sigma_0}{\sigma^2}\left(\sqrt{1 + \frac{2\Delta t^{-1}\sigma^2}{\delta^2}} + 1\right) - \frac{\mu_0}{\sigma_0}\right)\right\}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathsf{E}\left[\mathsf{FT}\left(z_0\right)\right] = {} & \Delta t + \frac{\sigma_0}{\delta\sqrt{2\pi}} e^{-\frac{\mu_0^2}{2\sigma_0^2}} \\
& + \left(\mathsf{FT}(0) - \Delta t + \frac{\mu_0}{\delta}\right) \Phi\left(\frac{\mu_0}{\sigma_0}\right) \\
& + \left(\mathsf{FT}(0) - \Delta t\right) e^{2\Delta t^{-1}\mathsf{FT}(0)\left(\frac{\sigma_0^2\delta^2}{\sigma^4} + \frac{\mu_0\delta}{\sigma^2}\right) + \frac{\Delta t^{-1}\sigma_0^2}{\sigma^2}} \Phi\left(-\frac{2\Delta t^{-1}\mathsf{FT}(0)\sigma_0\delta}{\sigma^2} - \frac{\mu_0}{\sigma_0}\right).
\end{aligned}$$