Welfare Analysis of Dark Pools

Krishnamurthy Iyer
School of Operations Research and Information Engineering
Cornell University
email: kriyer@cornell.edu

Ramesh Johari
Department of Management Science and Engineering
Stanford University
email: ramesh.johari@stanford.edu

Ciamac C. Moallemi
Graduate School of Business
Columbia University
email: ciamac@gsb.columbia.edu

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Abstract

We investigate the role of a class of alternative market structures known as electronic crossing networks or “dark pools”. Relative to traditional “lit” markets, dark pools offer investors the trade-off of reduced transaction costs in exchange for greater uncertainty of trade. Our paper studies the welfare implications of operating a dark pool alongside traditional lit markets. We study equilibria of a market with intrinsic traders and speculators, each endowed with heterogeneous fine-grained information, who endogenously choose between dark and lit venues. We establish that while dark pools attract relatively uninformed investors, the orders therein experience an implicit transaction cost in the form of adverse selection. On the other hand, we show that dark pools facilitate trade between relatively less informed intrinsic traders and speculators, inducing a positive liquidity effect on welfare. We study the interplay of these countervailing pressures on market welfare, and quantify regimes in which the welfare rises or falls upon the introduction of a dark pool.

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1. Introduction

Crossing networks, more commonly known as “dark pools,”\textsuperscript{1} are a market mechanism that seeks to directly facilitate trade between buyers and sellers of an asset outside of traditional “lit” exchanges (where the available liquidity is visible) or dealer markets. Dark pools operate by having buyers and sellers submit orders that are not visible to the overall market. Trades occur from direct matches between orders in the pool with complementary trading needs. One touted advantage of dark pools is that they can often directly match natural buyers with natural sellers of an asset without intermediating market makers as in a dealer market, or with market orders on an exchange. Thus, in the absence of transaction costs charged by liquidity providers, trade can be cheaper in a dark pool. On the other hand, trade is uncertain in a dark pool: \textit{a priori}, an agent faces a risk that their order will not be matched (or, “filled”), and they may face more disadvantageous prices at a later time. In this way, dark pools offer investors a choice that exposes a fundamental trade-off between \textit{price} and \textit{uncertainty of trade}.

To understand the trade-off between price and execution risk, a critical element is the role of \textit{information}, specifically, private, asymmetric information possessed by market participants as to the short-term value of an asset. The role of information in the analysis of market microstructure has been a longstanding topic of study, dating back to the seminal work of Glosten and Milgrom (1985) and Kyle (1985). In our setting, information is important for two fundamental reasons: First, if an agent has information or beliefs about short-term price changes, this will clearly impact their individual decision-making. For example, an agent might justify paying a premium for a certain trade given commensurate certainty in the value of the asset. Second, however, even in the absence of any such information, an agent needs to reason about the informational characteristics of those with whom they trade. In considering a contingent trade, agents with less information about the value of an asset face \textit{adverse selection} or a “winner’s curse”: by systematically trading with more informed investors, trade occurs in situations where it is in fact \textit{ex post} undesirable to the agent. Not surprisingly, informational considerations affect all aspects of market operation, from trader behavior to the profitability of market making. Given the complex strategic interactions involved, qualitative insights and operational guidelines can be elusive.

In this paper, we study the effect of asymmetric information on market welfare in the presence of a dark pool. Because of our focus on microstructure, we make two salient modeling choices. First, we focus on trading over relatively short timescales, and in particular on relatively short-term asymmetries of information among traders: even if long-term asymmetric information is present, it is unlikely to be consequential over the short term. By contrast, recent empirical evidence suggests that there is considerable and consequential variation in the level of short-term private information across traders (Moallemi et al., 2014). Second, we focus on a regime where a large number of relatively small traders interact. Over the short-term, even large trades are typically broken into many smaller child orders before being executed.\textsuperscript{2}

\textsuperscript{1}Strictly speaking, dark pools refer to market mechanisms where the available liquidity is not advertised or visible. Crossing networks are one type of dark pool based on the direct matching of buyers and sellers, often with reference to an external market to determine the price. We use the term “dark pool” colloquially to refer to this type of crossing network.

\textsuperscript{2}Note that one claimed benefit of dark pools is that they mitigate the pre-trade information leakage that might arise from the placement of a large order in the open market. In a typical dark pool, only trades that occur are reported to the broader market — the underlying orders are confidential — and these are only reported after the fact. In this paper, we do not focus on this aspect of dark pools. While the information leakage of a large execution is an important issue in electronic markets, this is typically managed by spreading out a large order into smaller orders across time (see, e.g., Moallemi et al., 2012), rather than executing such large orders on a dark pool. Indeed, in U.S. equity markets, order and trade sizes in a typical dark pool are largely comparable to those in lit markets (Tuttle, 2012).
Specifically, we consider a stylized, one period model, where a continuum of infinitesimal, risk-neutral agents can choose to trade an asset. Agents seeking to buy or sell an asset can do so on the open market. Here, they may trade the asset immediately and with certainty by trading with market-making intermediaries (i.e., a traditional, competitive dealer market), in exchange for paying a premium in the form of a transaction cost (i.e., the bid-offer spread). Alternatively, agents may choose to trade in a dark pool market where they are directly matched with other agents. Trade in the dark pool occurs by reference to the mid-market price in the open market, and hence does not incur a transaction cost. On the other hand, a dark pool presents a risky opportunity for trade: if there is a mismatch between the overall populations of buyers and sellers in a dark pool, then a subset of the agents’ orders will not be filled.

Agents make this decision based on private information as to the short-term future price of the asset (i.e., the common value or fundamental value), as well as their own intrinsic demand for the asset (i.e., their individual hedging demand or idiosyncratic value). A central feature of the model we study is a rich and high resolution model of information: all agents in our setting possess private information as to the asset value. However, and in contrast to their private signals are heterogeneous and vary in strength across agents.

Agents are also endowed with an idiosyncratic value for the asset. In this regard, there are three classes of agents: intrinsic buyers, with positive idiosyncratic value, intrinsic sellers, with negative idiosyncratic value, and speculators, with no idiosyncratic value — the latter class seek only to maximize their expected wealth. Note that speculators trade only on the basis of the private information they possess. Thus by varying the mass of speculators present in the market, our model admits exogenous variation of the level of information present in the market.

Our equilibrium concept involves a Bayes-Nash equilibrium among the agents, with transaction costs in the open market set through a zero-profit condition (i.e., competitive market makers). In our analysis, we compare two market structures: (1) one where agents choose between the dark pool and the open market; and (2) one where agents can trade only in the open market.

The main contribution of our paper is to leverage our model to study the relationship between asymmetric information and the welfare effects of the introduction of a dark pool. Intuitively, in our setting, monetary transfers are zero-sum when viewed in aggregate from a systemic perspective — the gain of one agent is the loss of another. Transfers of the asset, however, are not: the sale of the asset from an agent with low idiosyncratic value to an agent with high idiosyncratic value can be a Pareto improvement. Welfare quantifies such gains from trade.

We begin by discussing several structural properties of equilibrium in our model to provide context for our subsequent welfare analysis. First, we find that, in general, traders are segmented by their level of information. More precisely, in equilibrium, when all else is equal, the dark pool is utilized by relatively uninformed or mildly informed traders, whereas highly informed traders will trade in the open market so as to exploit their short-term information through guaranteed profitable executions. This implies that, via an information segmentation mechanism, trade in the dark pool will alter the informational characteristics of trade in the open market. In particular, the greater adverse selection risk will (all else being equal) cause spreads in the open market to widen. This suggests that information leakage from large orders is a second order effect in the choice of trade venue.
This finding is consistent with earlier literature on the effect of the introduction of a dark pool as well (see, e.g., [Zhu 2014]; [Hendershott and Mendelson 2000]).

In addition, in equilibrium in our model, traders in the dark pool experience adverse selection. Specifically, conditional on their order being filled in the dark pool, a buyer’s (resp., seller’s) expectation of the asset’s fundamental value is lower (resp., higher) than their prior, unconditional expectation. This arises from the fact that the execution of an order in the dark pool is correlated with the fundamental value of the asset in a way that is to the detriment of most dark pool participants: buyers in the dark pool are more likely to be filled when the dark pool price is above the fundamental value, while sellers are more likely to be filled when the dark pool price is below. The presence of speculators in the market further exacerbates this effect.

While adverse selection is also a feature of other models of crossing networks (e.g., [Zhu 2014]), the mechanism through which adverse selection arises in our model is different. In particular, this detrimental correlation cannot be explained by the traditional mechanism of information asymmetry: trade with a highly informed counterparty. Indeed, as mentioned earlier, the more informed traders trade in the open market. Instead, in our model, adverse selection is created through the aggregate behavior of the group of investors participating in the dark pool. These investors are all relatively uninformed, but in the case where the fundamental value is higher (resp., lower) than the dark pool transaction price, there are more investors with a slight positive (resp., negative) signal than the opposite. In other words, adverse selection endogenously occurs in the dark pool through the aggregation of diffuse information from a cross section of marginally informed agents.

We also note that traders on the same side as the price movement (e.g., buyers when the price is going up) experience “crowding out” in the dark pool (cf. [Hendershott and Mendelson 2000]). In our model this is essentially driven by the same mechanism as adverse selection: when the value of the security rises, more traders buy than sell. The resulting imbalance in the dark pool causes some buyers not to be filled — in particular, any intrinsic buyers who are crowded out cause a loss of welfare.

These effects — higher spreads in the open market, together with adverse selection and “crowding out” in the dark pool — appear on the surface to be negative forces on welfare. However, focusing only on those effects ignores the most salient effect of the introduction of a dark pool itself: in particular, by providing an alternative venue for trade, a dark pool can increase the overall volume of intrinsic traders that participate in the market. Note that the dark pool will draw in traders who are relatively less informed; both speculators and intrinsic traders. Therefore there is also a positive effect on welfare: namely, the introduction of the dark pool can make it possible for lesser informed intrinsic traders to cross with lesser informed speculators. Indeed, this effect is consistent with the “naive” view of the introduction of a dark pool: after all, traders have an additional venue for trade, and more choice should imply higher welfare. Taken together, these countervailing pressures on welfare interact with each other, and leave ambiguous both the eventual structure of equilibrium and the resulting prediction of whether welfare will rise or fall.

Our main results quantify regimes in which welfare rises or falls on the introduction of a dark pool. First, we consider a baseline model without speculators. In this setting, despite the fact that the introduction of the dark pool provides an alternative venue for trade, it does not induce a substantial new volume of intrinsic traders to participate; this is because without speculators, there is an insufficient mass of counterparties. Instead, the combination of higher transaction costs in the open market and relatively high adverse selection costs in the dark pool drive some intrinsic traders away, compared to a trading environment without the dark pool. This leads to a net welfare loss. We then show that the addition of speculators can mediate this effect, provided that the intrinsic traders have a moderately strong intrinsic value: in that regime, the dark pool does indeed function essentially as an alternative venue for trade, as in the naive intuition above. The result is a net
welfare gain. In the process, we provide characterization of how the relevant features of our model — notably, the intrinsic value of traders and the mass of speculators — relate to gains or losses in welfare.

In terms of testable implications, many of our model predictions are consistent with observations from the empirical literature on dark pool trading. The information segmentation of relatively uninformed trades to dark pools has been observed (Comerton-Forde and Putniņš 2015), nevertheless dark pool trades do have informational content (Nimalendran and Ray 2014) and there can be substantial adverse selection costs when trading in a dark pool (Næs and Odegaard 2006). Since we identify two parameter regimes, one where dark pool trading is welfare decreasing and the other is welfare increasing, a natural and important empirical question is to ask which regime real world financial markets fall under. As is the case with many of the theoretical models in the literature, our model is stylized, and many of the parameters (e.g., idiosyncratic values) cannot be directly calibrated. On the other hand, an increase in open market transaction costs has been observed in some studies (e.g., Comerton-Forde and Putniņš 2015, Degryse et al. 2014, Foley et al. 2012); this suggests at least one welfare-reducing effect is present.

1.1. Literature review

Our underlying open market setting here is reminiscent of the models of Copeland and Galai (1983) and Glosten and Milgrom (1985) for studying dealer markets with asymmetric information. In the area of crossing networks, the main reference is the work of Hendershott and Mendelson (2000), where a two-period model of a market with a dealer market and a crossing network populated with liquidity traders and informed traders is developed. The latter know the exact value of the security, face no strategic trade-off, and are thus assumed to always trade (in contrast to the present paper, where all traders act endogenously). The authors study the welfare of the liquidity traders in equilibrium, ignoring the welfare of the informed traders, and show that the crossing network impacts welfare through two competing externalities. First, there is a “liquidity” externality driven by stochastic imbalances between buyers and sellers that has a positive impact on liquidity. Second, there is a “crowding” externality, caused by traders overestimating the probability of an order getting filled in the dark pool, that has a negative impact on welfare. In the present paper, we have no stochastic imbalances and hence no liquidity externality. In the absence of a liquidity externality, there can only be a reduction in welfare in the Hendershott and Mendelson (2000) framework. In contrast, in our setting, all traders have heterogeneous, fine-grained but partial information about the value of the asset. Welfare effects are driven by informational imbalances, and can be positive or negative depending on the parameter regime.

Our model is also related to the work of Zhu (2014), who also builds an information-based framework for studying the relationship between an open market and a dark pool. However, there are key differences between our work and his; Zhu (2014) considers a coarse model of information where traders are either fully informed or completely uninformed, and seeks to understand the impact of a dark pool on incentives for the costly acquisition of information. Consistent with our results, he establishes that the dark pool attracts uninformed traders, and that this may reduce liquidity in the open market. Ultimately, however, Zhu (2014) focuses on the role of dark pools in price discovery, but without insight into the welfare implications. To contrast, our work does not speak to price discovery and instead focuses on welfare. Ye (2011) develops a model for deciding between an open market and a dark pool in the setting of Kyle (1985), but allows for only a single, fully informed trader with an endogenous choice of mechanism, and the model is limited by the fact that uninformed traders exogenously choose a venue.

A number of other authors develop theoretical models for dark pools in settings where either
the traders are uninformed, or all informed traders have symmetric information as to the asset value (e.g., Daniëls et al., 2013; Degryse et al., 2009; Afèche et al., 2014). Degryse et al. (2009) consider a dynamic setting with uninformed order flow on a dealer market and a crossing network, under varying informational settings about past order flow. Similar to our results, they find that the overall welfare may decrease on introducing a crossing network alongside a dealer market, unless the asset has high relative spread. Foster et al. (2007) consider a symmetric information setting with a specialist market, modeled as a continuous-auction market, and a volume-conditional crossing market, which clears only if there are sufficient orders. Under the right volume condition, the authors show that all traders in the market are weakly better off than in a setting with specialist market operating in isolation. Buti et al. (2014) model a dark pool that operates in parallel with a limit order book, and make welfare predictions in a symmetric information setting. Their model considers a trade-off between execution risk and cost that is similar to that in the present paper. However, as their paper is in a symmetric information setting, the authors pose it as a challenge to understand the impact of dark pools in the presence of asymmetric information, an important driver of real world financial markets. Our paper resolves this challenge, as our primary concern is to precisely understand how asymmetric information and adverse selection impact welfare in the presence of a dark pool.

Dark pools have also been studied in the optimal execution literature (e.g., Ganchev et al., 2010; Klöck et al., 2011; Kratz and Schöneborn, 2013, 2014), where the goal is to formulate and solve an individual agent’s decision problem of how to trade in order to efficiently liquidate a large portfolio. In such settings, however, the behavior of other agents in the market is described through non-strategic, reduced form specifications. Hence, such models are complementary to our work: their models are not meant to be used to reason about market structure counterfactuals.

2. Model

We study a single-period setting organized for trading shares of a single security. We assume two types of marketplaces exist to conduct trade: (1) an intermediated open market and (2) a dark pool market. A continuum of traders decide, based on their private information, whether to trade in the open market or to enter the dark pool. In this section, we describe the asset, the marketplaces, the intermediating market maker, the traders’ types, and their private signals, utilities, and strategies.

2.1. Asset

A single security is traded in the market at time $t = 0$. The common value of the security at $t = 1$ is unknown, and we model the uncertainty as a random variable $\sigma$. More specifically, we assume that $\sigma$ takes values in the set $\{-1, +1\}$, with either value equally likely. This captures, in a stylized manner, the notion that the security value may undergo either a positive or a negative jump of equal magnitude, and that the prior belief about this change in value is uninformed. We will shortly assume that all agents in the market are risk-neutral, and hence (ignoring idiosyncratic value and private information effects to be discussed shortly) the mid-market value of the security at time $t = 0$ is zero. We assume that $\sigma$ is fully revealed to all agents in the market at time $t = 1$.

2.2. Traders

The market has a continuum of infinitesimal traders, each seeking to buy or sell at most one share of the security at time $t = 0$. We assume that all of the traders are risk-neutral. This is motivated by the fact that our one period model takes place over a very short time horizon over
which each trader executes a small trade. In a setting with small trades over a short-time scale, utility functions are effectively linear.

Each trader $i$ is characterized by an idiosyncratic value $v_i$, that, along with the common value $\sigma$, determines the value the trader attaches to a single unit of the security. More precisely, we assume that, at time $t = 1$, a trader with idiosyncratic value $v_i$ values the security at $\sigma + v_i$. Here, the idiosyncratic value $v_i$ can capture, for example, the hedging demand particular to trader $i$. We assume that the initial position held by the trader is zero, and instead use the idiosyncratic value as a mechanism for non-informational trader heterogeneity. Indeed, since our traders are risk-neutral, a trader’s initial position is not relevant for decision making.

We fix a parameter $V > 0$ and we assume the idiosyncratic value $v_i$ can take one of three values: $+V$; $-V$; or zero. Our model admits and the structural results hold for more general distributions of the idiosyncratic value; we adopt this formulation to make the welfare analysis and the exposition simpler. In particular, depending on their idiosyncratic value, we characterize the traders into three groups: (1) intrinsic buyers, i.e., those traders with a positive idiosyncratic value ($v_i = +V$); (2) intrinsic sellers, i.e., those with a negative idiosyncratic value, ($v_i = -V$); and (3) speculators, i.e., those with a zero idiosyncratic value ($v_i = 0$).

Intuitively, in the absence of any private information, intrinsic buyers (resp., sellers) arrive at the market with the inclination to buy (resp., sell) one unit of the security. We emphasize that this initial inclination may get altered due to their private information (which we describe below). On the other hand, for a speculator, their motivation to trade arises only out of their private information. As a result, by varying the mass of speculators in the market (relative to the mass of intrinsic traders), we are able to parametrically vary the amount of information entering the market.

Finally, we assume that the mass or measure of intrinsic buyers and sellers is equal and normalized to 1, while the mass of speculators is $\mu \geq 0$. Thus, the total mass of traders in the market is $2 + \mu$.

2.3. Marketplaces

The market is composed of two distinct types marketplaces:

1. **Open market.** We envision the open market as a dealer market where, at time $t = 0$, any trader may enter an order to buy or sell a single unit of the security. The open market is intermediated by a market maker. We assume the market maker is risk-neutral, has no idiosyncratic value for the asset, and has an uninformed prior belief (vis-a-vis the private information of Section 2.4) on the common asset value $\sigma$. Therefore, the mid-market value of the asset to the market maker at time $t = 0$ is zero. The market maker charges an additional cost $\delta \in [0, 1]$ over this mid-market value in order to transact. In other words, at time $t = 0$, the security is bid at a price of $-\delta$ and offered at a price of $\delta$ in the open market. (One may also think of $2\delta$ as analogous to the bid-ask spread incurred by the market orders in specialist markets or limit order book markets.) At time $t = 0$, orders from all agents are simultaneously presented to and filled as a batch by the market maker (i.e., the market maker does not consider the orders sequentially or update the price). To summarize, the open market offers agents guaranteed immediate execution at time $t = 0$ at the cost of an additional transaction cost $\delta$.

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4Initially, in the partial equilibrium definition of Section 3.1, $\delta$ will be taken as an exogenous parameter. Later, in the competitive equilibrium of Section 3.2 $\delta$ will be determined endogenously.
2. **Dark pool.** At time $t = 0$, traders may choose to enter the dark pool to conduct trade by seeking to buy or sell a single unit of the security. The dark pool is a parallel marketplace without intermediating market makers. It clears trades an instant after time $t = 0$, at a price determined by reference to the prevailing mid-market price in the open market at time $t = 0$, i.e., a price of zero. Thus, a trade carried out at the dark pool market does not incur the transaction cost $\delta$ that is imposed in the open market.

The orders in the dark pool are cleared using a uniform random matching process between the two sides of the market, i.e., those traders seeking to buy and those traders seeking to sell. Each buy order has an equal chance of getting matched with one of the sell orders and vice versa. Thus, if the mass of buy orders in the dark pool is $m_b$ and the mass of sell orders is $m_s$, then when $m_b > m_s$, all sell orders get filled, while only $m_s$ of the buy orders get filled. Moreover, each sell order gets filled independently with probability $m_s/m_b$ — this quantity is known as the *fill rate* for buy orders. The situation is analogous when $m_s > m_b$.

Unless the mass of buy orders exactly equals the mass of sell orders, orders in the dark pool suffer from a risk of non-execution. Thus, a trader, while deciding between the open market and the dark pool, has to balance the trade-off between zero transaction cost and possibility of non-execution. We assume that any information regarding the orders or trades of either market mechanism is not revealed until time $t = 1$ (at which point it is moot).

Each trader in the market is strategic in selecting the marketplace to trade at. Each trader makes the entry and trade decisions based on her idiosyncratic value as well as her private information about $\sigma$. We next describe the private information structure among the traders.

### 2.4. Private information

Recall that the value of the security $\sigma$ at time $t = 1$ is an unknown random quantity at time $t = 0$. We assume that each trader $i$ at time $t = 0$ (before making any strategic decision) receives a private signal $s_i \in S$ informing her about the common value $\sigma$.

Formally, we assume that the price movement $\sigma$ is distributed according to a common prior $P$, where

$$P(\sigma = +1) = P(\sigma = -1) = 1/2. \quad (1)$$

Further, we assume a *conditionally independent* signal structure: conditional on the price movement $\sigma$, the signals $\{s_i\}$ are distributed independently and identically.\(^6\)

Since the only uncertainty in our model arises from the realization of the common value $\sigma$, which has a Bernoulli distribution, without loss of generality, we assume that each signal $s$ directly represents the posterior probability that $\sigma = 1$, and that the set of possible signals is the unit interval $S \equiv [0, 1]$. In other words, for all $s \in [0, 1]$,

$$P(\sigma = 1|s) = s. \quad (2)$$

Hence, a signal of $s = 1/2$ corresponds to a trader who is uninformed beyond the prior, while more informed traders would see signals closer to 0 or 1. As we will see in what follows, this

\(^5\)We assume that the acquisition of private information is costless.

\(^6\)To be precise, since we consider a model with a continuum of traders, a conditionally independent signal structure requires appealing to an exact law of large numbers argument (Qiao et al., 2014). We suppress these technical details for clarity.
representation of private information allows us to express traders’ utility for various actions as a linear function of their signals, thereby simplifying our analysis.

For tractability of analysis, we further assume that, given $\sigma$, the signals are distributed on $S = [0, 1]$ according to a distribution $F_\sigma$, with the following cumulative distribution function (cdf):

$$F_1(x) = x^2, \quad \text{and} \quad F_{-1} = 1 - (1 - x)^2, \quad \text{for all } x \in [0, 1].$$ \hspace{1cm} (3)

A simple application of Bayes’ rule verifies that, given the prior distribution (1), an observation of a signal from the distribution (3) yields the posterior belief (2).

Our model can be extended to a one-parameter family of signal distributions indexed by $\kappa > 0$, based on a sub-class of beta distributions. In particular, the signal distributions $F^{\kappa}_{\sigma}$ are given by

$$F^{\kappa}_{+} \sim \text{Beta}(\kappa + 1, \kappa), \quad \text{and} \quad F^{\kappa}_{-} \sim \text{Beta}(\kappa, \kappa + 1).$$

The quantity $\kappa$ parameterizes the informativeness of traders’ signal distribution in the following sense. As $\kappa \to \infty$, the signal distribution becomes more uninformative: $F^{\kappa}_{+}$ concentrates around 0.5 irrespective of the value of $\sigma$. On the other hand, as $\kappa \to 0$, the signal distribution becomes more informative: for small enough $\kappa > 0$, $F^{\kappa}_{+}$ is concentrated around 1, while $F^{\kappa}_{-}$ is concentrated around 0. Note that our choice of signal distribution in (3) corresponds to the case $\kappa = 1$. All our results in Section 4 continue to hold for general values of $\kappa$, while those in Section 5 require $\kappa = 1$.

2.5. Trader actions and utility

As the traders are risk-neutral, we assume that they seek to maximize the expected value of their position at $t = 1$ while deciding where and how to trade. We explain in detail below the decision problem faced by a trader at time $t = 0$.

At time $t = 0$, a trader with idiosyncratic value $v$ and private signal $s$, has to decide among the following actions:

1. $(OM, B)$: Buy a share at the open market, yielding an expected utility of

$$u_{OM,B}(v, s) = E[\sigma + v|s] - \delta = 2s - 1 + v - \delta.$$ \hspace{1cm} (4)

2. $(OM, S)$: Sell a share at the open market, yielding an expected utility of

$$u_{OM,S}(v, s) = -E[\sigma + v|s] - \delta = 1 - 2s - v - \delta.$$ \hspace{1cm} (5)

3. $(DP, B)$: Enter a buy order into the dark pool market, yielding an expected utility of

$$u_{DP,B}(v, s) = E[(\sigma + v)I\{\text{buy fill}\}|s],$$ \hspace{1cm} (6)

where $I\{\text{buy fill}\}$ denotes the indicator variable corresponding to the event that the buy order gets filled in the dark pool market.

4. $(DP, S)$: Enter a sell order into the dark pool market, yielding an expected utility of

$$u_{DP,S}(v, s) = -E[(\sigma + v)I\{\text{sell fill}\}|s],$$ \hspace{1cm} (7)

where $I\{\text{sell fill}\}$ denotes the indicator variable corresponding to the event that the sell order gets filled in the dark pool market.

\footnote{Here, we abuse the notation $F_\sigma$ to denote both the distribution as well as its cdf.}
5. \( N \): Do not trade, yielding an expected utility of \( u_N(v, s) = 0 \).

We let \( \mathcal{A} \triangleq \{(OM, B), (OM, S), (DP, B), (DP, S), N\} \) denote the action set available to a trader. Thus, a trader’s decision problem is to choose an action \( a \in \mathcal{A} \) that will maximize her expected utility, given her idiosyncratic value and her private signal:

\[
\max_{a \in \mathcal{A}} u_a(v, s).
\]

Observe that the expected utility of a trader on entering an order (buy or sell) into either market mechanism (open market or dark pool) depends on the actions of other traders in two ways: First, for orders in the open market, the expected utility depends on the transaction cost \( \delta \). In the competitive equilibrium definition of Section 3.2, this transaction cost will be determined endogenously, as a function of the collective behavior of all traders in the open market. Similarly, the expected utility of an order submitted into the dark pool depends on the likelihood of the order getting filled, which depends on how many other orders are present in the dark pool. This, in turn, depends on the actions of other traders in market. Thus, we see that the traders’ decision problems are intricately coupled and must be collectively determined in equilibrium.

2.6. Trader strategies

Having defined the traders’ decision problem, we now look at their strategies. A strategy for a trader specifies, for every signal \( s \in \mathcal{S} \), the action to take in the market. Formally, a strategy for a trader is a map \( \lambda : \mathcal{S} \to \mathcal{A} \). (Here and throughout the paper, we only consider pure strategies for the market participants.)

We will restrict our attention to the case where all traders with the same idiosyncratic value use the same strategy. In other words, we require the collection of strategies used by the traders to satisfy an anonymity condition, where the strategy used by a trader does not depend on her identity. With this condition in place, we denote the strategy of a trader with idiosyncratic value \( v \) as \( \lambda_v : \mathcal{S} \to \mathcal{A} \), and we define a strategy profile \( \lambda \triangleq (\lambda_{-V}, \lambda_0, \lambda_V) \) as the tuple of strategies employed by all the traders in the market.

2.7. Competitive market makers

We will be interested in studying market-making in the open market in a state of perfect competition with free entry. In such a situation, if the market maker made a positive expected profit, competitors would enter the open market and undercut the market maker by reducing the transaction costs. This suggests that in any equilibrium under perfect competition and free entry, the market maker’s profit should be zero.

Formally, we capture this competitive limit by assuming that the open market is organized by a risk-neutral market maker, with the uninformed prior \( P \), who receives the entire transaction cost \( \delta \) for each trade in the open market. Along with this revenue, she also faces the risk of adverse selection due to the presence of informed traders. The cost due to this risk depends on the traders’ strategies, which in turn depend on \( \delta \). We will then assume that the the transaction cost \( \delta \) in the open market is set so that the market maker’s total expected utility, as defined in (15), is zero.

3. Equilibrium

From the description of the market model in the preceding section, we observe that the traders’ decision problems are coupled due to the order matching process in the dark pool. In particular, a
3.1. Partial equilibrium

In this subsection we formulate a partial equilibrium concept for the game among the traders, given a fixed (and exogenously specified) transaction cost $\delta$ in the open market (in Section 3.2 we will return to this point, and determine $\delta$ endogenously). We first specify in more detail how a trader’s utility depends on the actions of all other traders in the market, and in particular, on the fill rate (i.e., the fraction of buy or sell orders that are executed) in the dark pool for each realization of the common value $\sigma$. We then introduce the appropriate equilibrium concept for the game among traders: Bayes-Nash equilibrium (BNE) (Tadelis, 2013). In a BNE, each trader’s strategy specifies for any signal $s$ an optimal action, holding fixed the other traders’ strategies.

3.1.1. Fill rates

In order to calculate the fill rates, as described in Section 2.3, we need to know the mass of buy and sell orders in the dark pool. These are determined by the strategies of the traders. Suppose the traders’ strategy profile is given by $\lambda = (\lambda_{-V}, \lambda_0, \lambda_V)$. For each idiosyncratic value $v \in \{0, \pm V\}$ and each action $a \in \mathcal{A}$, we define the set

$$\Theta_{v,a}^\lambda \equiv \{s \in [0, 1] : \lambda_v(s) = a\}. \quad (8)$$

Thus, $\Theta_{v,a}^\lambda \subset [0, 1]$ denotes the set of signal values for which a trader with idiosyncratic value $v$ chooses action $a$ under the strategy profile $\lambda$. In the rest of the paper, we restrict our attention to those strategies for which the preceding sets are $F_\sigma$-measurable.

Conditional on the security value $\sigma$, the mass of buy orders in the dark pool is given by

$$m_B^\lambda(\sigma) = \sum_{v \in \{\pm V\}} F_\sigma(\Theta_{v,\{\text{DP,B}\}}) + \mu F_\sigma(\Theta_{0,\{\text{DP,B}\}}). \quad (9)$$

Similarly, the mass of sell orders is given by

$$m_S^\lambda(\sigma) = \sum_{v \in \{\pm V\}} F_\sigma(\Theta_{v,\{\text{DP,S}\}}) + \mu F_\sigma(\Theta_{0,\{\text{DP,S}\}}). \quad (10)$$

Observe that $m_B^\lambda(1)$ is positive if and only if $m_B^\lambda(-1)$ is positive. Similarly, $m_S^\lambda(1)$ is positive if and only if $m_S^\lambda(-1)$ is. This follows from the fact that the measures $F_1$ and $F_{-1}$ are equivalent.

Now, suppose both $m_B^\lambda(\sigma)$ and $m_S^\lambda(\sigma)$ are positive. As a buy order in the dark pool is matched uniformly with a sell order, if $m_B^\lambda(\sigma) \leq m_S^\lambda(\sigma)$, a buy order in the dark pool gets filled with certainty, while a sell order gets filled with probability $m_B^\lambda(\sigma)/m_S^\lambda(\sigma)$. On the other hand, if $m_B^\lambda(\sigma) \geq m_S^\lambda(\sigma)$, a sell order gets filled with certainty, while a buy order gets filled with probability $m_S^\lambda(\sigma)/m_B^\lambda(\sigma)$. Thus for every value of $\sigma$, we can define the buy fill rate $\phi_B^\lambda(\sigma)$ and sell fill rate $\phi_S^\lambda(\sigma)$ as

$$\phi_B^\lambda(\sigma) \equiv \mathbb{E}[I\{\text{buy fill}\} | \sigma] = \min \left(1, \frac{m_S^\lambda(\sigma)}{m_B^\lambda(\sigma)}\right),$$

$$\phi_S^\lambda(\sigma) \equiv \mathbb{E}[I\{\text{sell fill}\} | \sigma] = \min \left(1, \frac{m_B^\lambda(\sigma)}{m_S^\lambda(\sigma)}\right), \quad (11)$$

The trader’s optimal action depends on how many other traders enter orders in the dark pool as well as the level of the transaction cost $\delta$ in the open market. For any fixed transaction cost $\delta$, this defines a game among the traders, for which we define a partial equilibrium concept. As described in the preceding section, the transaction cost $\delta$ is subsequently set so that the market maker’s expected utility is zero; we call a resulting solution a competitive equilibrium. In this section we formalize the definitions of both partial and competitive equilibria.
where \( m_B^\lambda(\sigma) \) and \( m_S^\lambda(\sigma) \) are as defined in (9) and (10). For completeness, we set \( \phi_B^\lambda(\sigma) = 0 \) if \( m_B^\lambda(\sigma) = 0 \) and we set \( \phi_S^\lambda(\sigma) = 0 \) if \( m_S^\lambda(\sigma) = 0 \).

### 3.1.2. Traders’ expected utility

From the fill rates, we obtain refined expressions for a trader’s utility, in terms of other traders’ actions. Suppose the strategy profile in the market is given by \( \lambda = (\lambda_{V}, \lambda_{0}, \lambda_{V}) \). In order to explicitly specify the dependence of the trader’s utility on the strategy profile, we qualify them with a superscript \( \lambda \). Then, from (6), we have

\[
\begin{align*}
    u_{DP,B}^\lambda(v,s) &= E[(\sigma + v)I\{\text{buy fill}\}|s] \\
    &= E[E[(\sigma + v)I\{\text{buy fill}\}][\sigma]|s] \\
    &= E[(\sigma + v)\phi_B^\lambda(\sigma)|s] \\
    &= s(1 + v)\phi_B^\lambda(1) + (1 - s)(-1 + v)\phi_B^\lambda(-1). \\
\end{align*}
\]

(12)

Here, we have used the tower property of conditional expectation in the second equality. Similarly, from (7), we have

\[
\begin{align*}
    u_{DP,S}^\lambda(v,s) &= -E[(\sigma + v)I\{\text{sell fill}\}|s] \\
    &= -E[E[(\sigma + v)I\{\text{sell fill}\}][\sigma]|s] \\
    &= -s(1 + v)\phi_S^\lambda(1) - (1 - s)(-1 + v)\phi_S^\lambda(-1). \\
\end{align*}
\]

(13)

Finally, observe that, in the open market, a trader transacts directly with the market maker. Thus, the utilities of actions involving trade in the open market (or the explicit choice not to trade) do not depend on the actions of other traders or on the strategy profile \( \lambda \). Hence, we have, as in (4)–(5),

\[
\begin{align*}
    u_{OM,B}^\lambda(v,s) &= 2s - 1 + v - \delta, \\
    u_{OM,S}^\lambda(v,s) &= 1 - 2s - v - \delta, \\
    u_N^\lambda(v,s) &= 0.
\end{align*}
\]

for all \( v \in \{0, \pm V\} \) and \( s \in [0,1] \).

### 3.1.3. Bayes-Nash equilibrium

We use Bayes-Nash equilibrium as the partial equilibrium solution concept for the game among traders, for a fixed transaction cost \( \delta \).

**Definition 1 (Partial equilibrium).** A strategy profile \( \lambda = (\lambda_{V}, \lambda_{0}, \lambda_{V}) \) and a fixed open market transaction cost \( \delta \) together constitute a partial equilibrium if, assuming the open market transaction cost is fixed at \( \delta \), the strategy profile \( \lambda \) satisfies the Bayes-Nash equilibrium (BNE) condition given by

\[
\lambda_v(s) \in \arg\max_{a \in A} u^\lambda_a(v,s),
\]

(14)

for all \( v \in \{0, \pm V\} \) and \( s \in [0,1] \).

Thus, in a partial equilibrium, each trader’s strategy employs an optimal action for every signal, fixing all other traders’ strategies and assuming a given open market transaction cost.

In what follows, it will be useful to adopt the following convention breaking ties among various actions: \( N > (DP,B) > (DP,S) > (OM,B) > (OM,S) \). This is arbitrary, and is done primarily so that there is a unique best response at each signal for any trader. (As will be seen later, tie-breaking
is only needed on a set of measure zero, and hence does not alter fundamental characteristics of a
strategy profile such as the associated fill rates in the dark pool or the adverse selection experienced
by the market maker.) With the tie-breaking rule in place, the inclusion in the definition of a partial
equilibrium can be replaced with an equality. Further, given any strategy profile $\lambda$, we can now
define a best response strategy profile $\Lambda[\lambda] \triangleq (\Lambda_{-V}[\lambda], \Lambda_0[\lambda], \Lambda_V[\lambda])$, where for any $v \in \{0, \pm V\}$, $\Lambda_v[\lambda]$ is the unique strategy defined by

$$
\Lambda_v[\lambda](s) \triangleq \arg\max_{a \in A} u^\lambda_{a}(v, s),
$$

for all $s \in [0, 1]$. The definition of a partial equilibrium $(\lambda, \delta)$ can now be simplified to require that
the strategy profile $\lambda$ satisfy $\lambda = \Lambda[\lambda]$, i.e., a fixed point of the best response map $\Lambda$ assuming
a transaction cost $\delta$. Note that the map $\Lambda$ implicitly depends on the mass of speculators $\mu$, the
idiosyncratic value $V$, and the transaction charge $\delta$; we will make these dependencies explicit when
the context demands.

### 3.2. Competitive equilibrium

In this subsection we define a competitive equilibrium by adding the zero-profit condition for
the market maker to the preceding definition of a partial equilibrium. Specifically, in a competitive
equilibrium, the value of the transaction cost $\delta$ will be endogenously determined through a zero-
profit condition. We start by fixing the transaction cost $\delta$, and compute the expected utility of
the market maker given a partial equilibrium among the traders. We then use this calculation to
formalize the zero-profit condition and determine the level of the transaction cost $\delta$.

#### 3.2.1. Market maker’s expected utility

Suppose the transaction cost in the open market is $\delta$, and consider a partial equilibrium $(\lambda, \delta)$. Let $m^\lambda_{(OM,B)}(\sigma)$ (resp., $m^\lambda_{(OM,S)}(\sigma)$) denote the volume of buy orders (resp., sell orders) in the
open market, conditional on the security value $\sigma$. As the market maker receives the transaction
cost on each trade in the open market, the total revenue to the market maker is the product of
the transaction cost $\delta$ and the total volume of trade in the open market. Thus, the total expected
revenue for the market maker due to the transaction cost is given by

$$
u_{tr}(\delta, \lambda) = \delta E \left[ m^\lambda_{(OM,B)}(\sigma) + m^\lambda_{(OM,S)}(\sigma) \right].
$$

Next, note that conditional on $\sigma$, the market maker’s net position in the security is given by the
difference $m^\lambda_{(OM,S)}(\sigma) - m^\lambda_{(OM,B)}(\sigma)$. Ignoring the transaction cost, the expected mark-to-market
profit from this position between time $t = 0$ and $t = 1$ is given by

$$
u_{mm}(\delta, \lambda) = E \left[ \sigma \left( m^\lambda_{(OM,S)}(\sigma) - m^\lambda_{(OM,B)}(\sigma) \right) \right].
$$

Observe that if the net quantity of the security bought by the market maker had been independent
of the security value $\sigma$, then in expectation this mark-to-market profit would have value zero. In
general, however, if according to the strategy profile $\lambda$, only the more informed traders choose to
trade with the market maker, it will likely be the case that $u_{mm}(\delta, \lambda) \leq 0$. Indeed, $-u_{mm}(\delta, \lambda)$
represents the cost incurred by the market maker due to adverse selection.

Putting these together, the total expected utility of the market maker is

$$
u_M(\delta, \lambda) \triangleq \nu_{tr}(\delta, \lambda) + u_{mm}(\delta, \lambda)
= E \left[ (\delta - \sigma)m^\lambda_{(OM,B)}(\sigma) + (\delta + \sigma)m^\lambda_{(OM,S)}(\sigma) \right].
$$

(15)
3.2.2. Zero-profit condition

We are now ready to formally define a competitive equilibrium, which combines our BNE conditions with a zero-profit condition for the market maker:

**Definition 2.** A strategy profile $\lambda$ and a transaction cost $\delta$ together constitute a competitive equilibrium if and only if

1. For a transaction cost of $\delta$, the strategy profile $\lambda$ constitutes a BNE; and
2. Given the strategy profile $\lambda$ and transaction cost $\delta$, the market maker’s utility $u_M(\delta, \lambda)$, as defined in (15), is zero.

Observe that, from (15),

$$u_M(\delta, \lambda) \geq E\left[ (\delta - |\sigma|) \left( m^\lambda_{(OM,B)}(\sigma) + m^\lambda_{(OM,S)}(\sigma) \right) \right].$$

Then, if $\delta > |\sigma| = 1$ and there is any trade in the open market, the market maker’s utility will be strictly positive. Therefore, when the transaction cost is large ($\delta > 1$), there can be no competitive equilibrium involving trade in the open market. To avoid this uninteresting situation, we require that $\delta \in [0, 1]$.

4. Structural results on partial equilibrium

In this section, we introduce a natural symmetry condition on the equilibrium strategies that lets us obtain results on the structure of a partial equilibrium. First, we show that partial equilibria involving such strategies are completely specified by the buy fill rate in the dark pool. Second, from the structure of the equilibrium strategy, we show how the traders’ choice of the trading venue depends on their private information. Finally, we demonstrate that, in equilibrium, trade in the dark pool experiences adverse selection.

4.1. Symmetric strategy profiles

As the model we consider is symmetric with respect to change in the asset value $\sigma$, the idiosyncratic valuations of intrinsic buyers and sellers, and the mass of intrinsic buyers or sellers in the market, we focus our attention on a class of strategy profiles that satisfy a natural symmetry requirement. We begin with the following definition:

**Definition 3.** A strategy profile $\lambda = (\lambda_{-V}, \lambda_0, \lambda_V)$ is symmetric if the following holds, for all $v \in \{0, \pm V\}$ and $s \in [0, 1]$:

$$\begin{align*}
\lambda_v(s) &= (OM, B) \quad \text{if and only if} \quad \lambda_{-v}(1-s) = (OM, S), \\
\lambda_v(s) &= (DP, B) \quad \text{if and only if} \quad \lambda_{-v}(1-s) = (DP, S), \\
\lambda_v(s) &= N \quad \text{if and only if} \quad \lambda_{-v}(1-s) = N.
\end{align*}$$

In words, in a symmetric strategy profile, we require that if an intrinsic buyer with signal $s$ enters an order into the dark pool market, then an intrinsic seller with signal $1-s$ enters an opposite order into the dark pool market. Similarly, if an intrinsic buyer with signal $s$ trades in the open market, then an intrinsic seller with signal $1-s$ trades in the open market in the opposite direction. Finally, we require analogous conditions to hold for speculators with signals $s$ and $1-s$.

As our first result shows, the set of symmetric strategy profiles is closed under the best response map. The proof is deferred to the appendix.
Lemma 1. Suppose \( \lambda = (\lambda_-, \lambda_0, \lambda_V) \) is a symmetric strategy profile. Then the best response strategy profile \( \Lambda(\lambda) \) is symmetric.

The preceding result implies that we can restrict ourselves to the set of symmetric strategy profiles in order to identify a partial equilibrium. This is useful because, as we show next, in any partial equilibrium with symmetric strategy profile, the fill rates are symmetric and can be characterized by a single parameter.

4.2. Fill rates

Observe that, following Definition 3 in a symmetric strategy profile \( \lambda \), for any \( v \in \{0, \pm V\} \) and any \( s \in [0, 1] \), we have \( s \in \Theta^\lambda_{v,(DP,B)} \) if and only if \( 1 - s \in \Theta^-_{v,(DP,S)} \), where \( \Theta^\lambda_{v,(DP,B)} \) and \( \Theta^-_{v,(DP,S)} \) are the sets of signals resulting in buy and sell orders in the dark pool from a trader with idiosyncratic value \( v \), as defined in (3). Thus, from (3), we obtain for \( \sigma \in \{-1, 1\} \),

\[
m^\lambda_B(\sigma) = \sum_{v \in \{\pm V\}} F_s(\Theta^\lambda_{v,(DP,B)}) + \mu F_s(\Theta^\lambda_{0,(DP,B)}) = \sum_{v \in \{\pm V\}} F_s(1 - \Theta^-_{v,(DP,S)}) + \mu F_s(1 - \Theta^\lambda_{0,(DP,S)}) = \sum_{v \in \{\pm V\}} F^-_s(\Theta^-_{v,(DP,S)}) + \mu F^-_s(\Theta^\lambda_{0,(DP,S)}) = m^\lambda_S(-\sigma).
\]

Here, in the second line, we have defined \( 1 - A \triangleq \{1 - x : x \in A\} \) for any set \( A \subset \mathbb{R} \). The equality in the third line follows from the symmetry of the signal structure. In particular, we have \( f_1(s) = f_{-1}(1 - s) \) for all \( s \in [0, 1] \), where \( f_\sigma \) is the density of \( F_\sigma \) for \( \sigma \in \{-1, +1\} \). The last equality follows from (10).

Thus, when the strategy profile is symmetric and \( m^\lambda_B(\sigma) \) is positive for all \( \sigma \), the fill rates, as defined in (11), satisfy

\[
\phi^\lambda_B(\sigma) = \phi^\lambda_S(-\sigma) = \min \left(1, \frac{m^\lambda_B(\sigma)}{m^\lambda_S(\sigma)} \right) > 0, \quad \text{for } \sigma \in \{-1, 1\}.
\]

This implies that, when \( m^\lambda_B(\sigma) \) is positive for all \( \sigma \) and \( \lambda \) is symmetric, the fill rates must satisfy one of the following two possibilities:

\[
\phi^\lambda_B(1) = \phi^\lambda_S(-1) = 1, \quad 0 < \phi^\lambda_B(-1) = \phi^\lambda_S(1) < 1; \quad \text{OR (17)}
\]

\[
0 < \phi^\lambda_B(1) = \phi^\lambda_S(-1) \leq 1, \quad \phi^\lambda_B(-1) = \phi^\lambda_S(1) = 1. \quad (18)
\]

In the former case, a buy order in the dark pool is more likely to be filled when the asset value increases, and less likely when it decreases. On the other hand, in the latter case, a buy order in the dark pool is less likely to be filled when the asset value increases, and more likely when it decreases. The following result shows that, in any partial equilibrium with trade in the dark pool, the fill rates must satisfy the latter condition. The proof is deferred until the appendix.

Theorem 1. For any \( V > 0 \), \( \delta \in [0, 1] \), and \( \mu \geq 0 \), suppose there is a partial equilibrium \((\lambda, \delta)\) with symmetric strategy profile \( \lambda \) involving trade in the dark pool.\(^8\) Then,

\[
0 < \phi^\lambda_B(1) = \phi^\lambda_S(-1) \leq 1, \quad \phi^\lambda_B(-1) = \phi^\lambda_S(1) = 1.
\]

\(^8\)Note, if the equilibrium symmetric strategy profile \( \lambda \) does not involve trade in the dark pool, then the fill rates satisfy \( \phi^\lambda_B(\sigma) = \phi^\lambda_S(\sigma) = 0 \) by definition.
The preceding result establishes that in any partial equilibrium in symmetric strategies with trade in the dark pool, a buy order in the dark pool will get filled with certainty if the asset value is low. As we show later in Section 4.5, this aspect of a partial equilibrium plays an important role in generating an adverse selection cost that is incurred by traders in the dark pool.

4.3. Threshold strategies

Next, we show that the strategies in a partial equilibrium take a particularly simple form: there are thresholds that completely determine behavior of intrinsic traders and speculators. Moreover, these thresholds can be obtained by identifying those signal values where two (or possibly more) actions yield the same expected utility for a trader. Formally, for each transaction cost \( \delta \in [0, 1] \), fill rate \( f \in [0, 1] \) and \( v \in \mathbb{R} \), define the following sub-intervals\(^9\) of the unit interval \([0, 1]\):

\[
\begin{align*}
\Theta_{v,(OM,B)}^{f,\delta} & \triangleq \begin{cases} 
\left(\max\left\{\frac{1-v+\delta}{2v}, \frac{\delta}{(1+v)(1-f)}\right\}, 1\right], & \text{if } v \in [0, 1]; \\
(1-v+\delta, 1], & \text{if } v > 1 \text{ and } f > 0; \\
\left(\frac{1-v+\delta}{2}, 1\right] \cap [0, 1], & \text{if } v > 1 \text{ and } f = 0; \\
1 - \Theta_{v,(OM,S)}^{f,\delta}, & \text{if } v < 0,
\end{cases} \\
\Theta_{v,(DP,B)}^{f,\delta} & \triangleq \begin{cases} 
\left(\min\left\{\frac{f(1-v)}{(1-f)(1+v)}, 1\right\}, 1\right], & \text{if } v \in [0, 1]; \\
\left(0, \min\left\{\frac{\delta}{(1+v)(1-f)}, 1\right\}\right], & \text{if } v > 1 \text{ and } f > 0; \\
\emptyset, & \text{if } v > 1 \text{ and } f = 0; \\
1 - \Theta_{v,(DP,S)}^{f,\delta}, & \text{if } v < 0,
\end{cases} \\
\Theta_{v,N}^{f,\delta} & \triangleq \begin{cases} 
\left[\max\left\{\frac{1-v-\delta}{2(1-v+f(1+v))}, \frac{1-v}{1-v+f(1+v)}, \frac{1-v+\delta}{2}\right\}, 1\right], & \text{if } v \in [0, 1]; \\
\emptyset, & \text{if } v > 1 \text{ and } f > 0; \\
\left[0, \frac{1-v+\delta}{2}\right], & \text{if } v > 1 \text{ and } f = 0; \\
1 - \Theta_{v,N}^{f,\delta}, & \text{if } v < 0,
\end{cases} \\
\Theta_{v,(DP,S)}^{f,\delta} & \triangleq \begin{cases} 
\left[\max\left\{1 - \frac{\delta}{(1-v)(1-f)}, 0\right\}, \frac{f(1-v)}{f(1-v)+1+v}\right], & \text{if } v \in [0, 1]; \\
\emptyset, & \text{if } v > 1; \\
1 - \Theta_{v,(DP,B)}^{f,\delta}, & \text{if } v < 0,
\end{cases} \\
\Theta_{v,(OM,S)}^{f,\delta} & \triangleq \begin{cases} 
\left[0, \min\left\{\frac{1-v-\delta}{2}, 1 - \frac{\delta}{(1-v)(1-f)}\right\}\right], & \text{if } v \in [0, 1]; \\
\emptyset, & \text{if } v > 1; \\
1 - \Theta_{v,(OM,B)}^{f,\delta}, & \text{if } v < 0.
\end{cases}
\end{align*}
\]

Here, for any set \( A \), we define \( 1 - A \triangleq \{1 - x : x \in A\} \). For each \( \delta \in [0, 1] \), define \( \mathcal{H}(\delta) \) as a one parameter family of symmetric strategy profiles

\[
\mathcal{H}(\delta) \triangleq \left\{ \lambda_{f,\delta}^{\delta} : f \in [0, 1] \right\},
\]

where for any \( f \in [0, 1] \), and \( v \in \{0, \pm V\} \), we define

\[
\lambda_{v,a}^{f,\delta}(s) \triangleq a, \quad \text{if } a \in \mathcal{A}, \ s \in \Theta_{v,a}^{f,\delta};
\]

\(^9\)We adopt the convention that an interval is defined to be the empty set if its endpoints are not ordered, i.e., \([a, b] \triangleq \emptyset\) if \( a > b \); while \((a, b) \triangleq \emptyset\), \((a, b) \triangleq \emptyset\), and \([a, b] \triangleq \emptyset\) if \( a \geq b \). Furthermore, we define \( 1/0 = \infty \) and \( 0/0 = 1 \).
In words, the strategy $\lambda_{f,\delta}^v$ picks the action for a given signal according to membership in the intervals defined in (19). In the case where one of the intervals in (19) is empty, the corresponding action is not used. In this way, intervals defined in (19) partition the unit interval into disjoint segments over which different actions are taken, as illustrated in Figure 1.

We refer to $\mathcal{H}(\delta)$ as the set of threshold strategies, since, for a fixed idiosyncratic value $v$, these strategies determine actions through a set of thresholds (the endpoints of the intervals defined above) that partition the set of possible signals. The following lemma establishes the importance of these strategies in equilibrium. The proof is deferred to the appendix.

**Lemma 2.** Suppose $\delta \in [0, 1]$, $\mu \geq 0$, and $V > 0$. Then,

(i) For any $f \in [0, 1]$ and $v \in \{0, \pm V\}$ the intervals $\{\Theta_{v,a}^{f,\delta}\}_{a \in \mathcal{A}}$ form a partition of the signal set $S = [0, 1]$, i.e., they are mutually exclusive and collectively exhaustive.

(ii) Suppose $\lambda$ is a symmetric strategy profile with fill rates $\phi_B^\lambda(1) \triangleq f \in (0, 1]$ and $\phi_S^\lambda(1) = 1$, then $\Lambda[\lambda] = \lambda^{f,\delta} \in \mathcal{H}(\delta)$. For a symmetric strategy profile $\lambda$ with fill rates $\phi_B^\lambda(1) = \phi_S^\lambda(1) = 0$, we have $\Lambda[\lambda] = \lambda^{0,\delta} \in \mathcal{H}(\delta)$.

Part (i) simply establishes that the threshold strategies described above are well-defined. Part (ii) has two important implications. First, observe that Theorem 1 guarantees that any partial equilibrium with a symmetric strategy profile satisfies the hypotheses of part (ii). Therefore, part (ii) guarantees that in any such partial equilibria, the traders employ threshold strategies. Second, observe that, for a fixed $\delta$, a threshold strategies is uniquely characterized by the parameter $f \in [0, 1]$. Then, combined with first implication, in any partial equilibrium, the Bayes-Nash fixed point condition takes the form $\Lambda[\lambda^{f,\delta}] = \lambda^{f,\delta}$. This is a one parameter fixed point equation involving a single unknown, the buy fill rate $f$. The second implication turns out to be crucial for the downstream analysis of the market in Section 5 as well as in numerical analysis through the use of line search methods for equilibrium computations; we discuss this in detail in Section 6.1.

Before continuing, we note that, for any transaction cost $\delta \in [0, 1]$ there always exists a trivial partial equilibrium where no trader enters the dark pool. This corresponds to $\lambda^{0,\delta} \in \mathcal{H}(\delta)$. This arises due to the fact that trade in the dark pool occurs only through matching. In particular, from a single trader’s point of view, if no other trader enters the dark pool, then the fill rate is zero, and hence there is no benefit to unilaterally deviating to enter the dark pool. Observe that this is the same outcome that arises in a market without the presence of a dark pool. Subsequently, when evaluating how the presence of a dark pool affects the market, we will compare market characteristics in this equilibrium with those in an equilibrium where there is trade in the dark pool.

Further, a partial equilibrium with trade in the dark pool may not exist for certain values of the model parameters. Typically, such a partial equilibrium may not exist if the traders’ idiosyncratic
values are small, there are too many speculators, or if the transaction cost in the open market is too low. Intuitively, in the former two scenarios, if trade were to occur in the dark pool, it would primarily be based on the differences in traders’ private information. However, such a situation is precluded by the no-trade theorem of Milgrom and Stokey (1982), which prohibits any trade among rational (risk-neutral) traders who differ only in their beliefs. In the latter scenario of low transaction cost in the open market, the traders may prefer to trade with certainty with the market maker in the open market over entering orders in the dark pool.

4.4. Information segmentation

As established in Section 4.3, in any partial equilibrium with a symmetric strategy profile, the traders must employ threshold strategies. In such strategies, traders with equal idiosyncratic value are segmented according to their private information in order to determine actions. This segmentation creates a relationship between a trader’s choice of venue to trade and their informedness and takes a particular form, as evidenced in the following corollary to Lemma 2. The proof is straightforward, and we omit it for brevity.

**Corollary 1 (Information segmentation).** Define the following total order on the action set \( A \),

\[
(OM, S) \prec (DP, S) \prec N \prec (DP, B) \prec (OM, B).
\]

Given a transaction cost \( \delta \in [0, 1] \), suppose there exists a partial equilibrium with symmetric strategy profile \( \lambda \). Then, for all \( v \in \{0, \pm V\} \) and \( 0 \leq s \leq s' \leq 1 \), we have that \( \lambda_v(s) \preceq \lambda_v(s') \).

Corollary 1 is depicted pictorially in Figure 1. It establishes that a particular ordering must hold in the actions chosen in all partial equilibria. Two observations can be made about this ordering:

1. Regarding the direction of trade, observe that for a fixed idiosyncratic value \( v \), if a trader with a signal \( s \) chooses to sell, all traders with signals less than \( s \) will also sell. Similarly, if a trader with a signal \( s \) chooses to buy, all traders with signals greater than \( s \) will also buy. Loosely speaking, traders with high signals will buy, while traders with low signals will sell, all else being equal.

2. Regarding the choice of venue, observe that for a fixed idiosyncratic value \( v \), if a trader with a signal \( s \) chooses to sell (resp., buy) in the open market, all traders with signals less (resp., greater) than \( s \) will also sell (resp., buy) in the open market. Loosely speaking, traders who are more informed (i.e., with signals closer to 0 or 1 depending on the direction of trade) will prefer the open market to the dark pool. In other words, the dark pool will be populated with traders who are relatively less informed than those who choose the open market.

The latter observation suggests that, via an information segmentation mechanism, trade in the dark pool will alter the informational characteristics of trade in the open market. This fact has important downstream implications for transactions costs in the open market in competitive equilibrium, which we will see in Section 5.

4.5. Adverse selection

Consider the following definition:

**Definition 4 (Adverse selection).** A trader submitting a buy (resp., sell) order in the dark pool suffers from adverse selection if her expectation of the value of the asset, conditional on the order
being filled, is lower (resp., higher) than her (unconditional) expectation of the value of the asset. Formally, we have for any \( s \in [0, 1] \),

\[ E[\sigma|s, \text{buy fill}] \leq E[\sigma|s] \leq E[\sigma|s, \text{sell fill}]. \]

The following result, whose proof is deferred until the appendix, states that adverse selection in the dark pool is pervasive in any partial equilibrium:

**Theorem 2.** In any partial equilibrium \( \lambda \) involving symmetric strategies with trade in the dark pool, all traders in the dark pool suffer from adverse selection. Defining the adverse selection cost

\[ \text{Adv}(\lambda) := E[\sigma|s = 1/2] - E[\sigma|s = 1/2, \text{buy fill}], \]

we have \( \text{Adv}(\lambda) = E[\sigma|s = 1/2, \text{sell fill}] - E[\sigma|s = 1/2] \geq 0 \). Furthermore, the adverse selection cost under such an equilibrium is strictly decreasing with the buy (sell) fill rate.

Adverse selection arises from the fact that the execution of an order in the dark pool is correlated with the movement in the value of the asset in a detrimental way: a buy (resp., sell) order is more likely to be executed when the value of the asset moves down (resp., up). However, this detrimental correlation cannot be directly attributed to information asymmetry in the dark pool; in fact, as mentioned earlier, the more informed traders trade in the open market. Rather, adverse selection is created through the aggregate behavior of the group of overall traders participating in the dark pool. In particular, consider the behavior of a speculator \( (v = 0) \). In a symmetric equilibrium, by Lemma 2, the subset of signals \( \Theta_{f,\delta,0}^{(DP,B)} \subset [0, 1] \) for which a speculator chooses to buy in the dark pool and the subset of signals \( \Theta_{f,\delta,0}^{(DP,S)} \subset [0, 1] \) for which a speculator chooses to sell in the dark pool are sub-intervals of the real line of equal length, and are symmetric about the point \( s = 1/2 \). In the case where \( \sigma = +1 \) (resp., \( \sigma = -1 \)), then, clearly the probability mass of speculators who choose to buy (resp., sell) in the dark pool is larger than those who choose to sell (resp., buy) in the dark pool. This mismatch of masses creates adverse selection amongst speculators in the dark pool, and this intuition extends as well to intrinsic buyers and sellers. In this way, adverse selection endogenously occurs in the dark pool through the aggregation of diffuse information from a cross section of marginally informed agents.

Thus, adverse selection is an intrinsic characteristic of any partial equilibrium in the market. Note that this adverse selection imposes an implicit transaction cost on the trader in the dark pool. This explains why there may not be a partial equilibrium involving trade in the dark pool for sufficiently low values of the (explicit) transaction cost in the open market.

### 5. Welfare analysis of competitive equilibria

In this section we exploit our structural understanding of partial equilibria to analyze competitive equilibria. Our emphasis is on understanding the welfare consequences of the introduction of a dark pool. As observed in the preceding section, the trades in the dark pool suffer from adverse selection. However, at the same time, the introduction of a dark pool offers participants greater opportunities for trade than before. Consequently, when traders and market makers are strategic and adapt to the presence of the dark pool, it is not \( a \ priori \) evident whether its introduction increases or decreases the overall welfare of the market.

Before we begin, we clarify the definition of welfare in our model. As usual, we define the welfare \( w(\delta, \lambda) \) of a competitive equilibrium \((\delta, \lambda)\) to be the sum of the expected utility of all the
participants (the traders and the market maker) in the market. Formally, we have

\[ w(\delta, \lambda) \triangleq \sum_{a \in A} \sum_{v \in \{-V, +V\}} \mathbb{E} \left[ u_{\lambda}^a(v, s) I\{\lambda_v(s) = a\} \right] + \mu \sum_{a \in A} \mathbb{E} \left[ u_{\lambda}^a(0, s) I\{\lambda_0(s) = a\} \right] + u_M(\delta, \lambda), \]

where the \( u_{\lambda}^a(v, s) \) denotes the expected utility of a trader with idiosyncratic value \( v \) and signal \( s \) taking an action \( a \) (as defined in (4), (5), (12), and (13)), and \( u_M(\delta, \lambda) \) denotes the expected utility of the market maker (as defined in (15)). (Here the expectation is over the signal \( s \), and is with respect to the uninformed common prior \( P \).)

Since all agents in the market are risk-neutral, it follows that the monetary transfers among the agents in the market do not affect the welfare. Hence, the welfare in the market depends solely on the final allocation of the security in the market. In particular, the market welfare is higher if more intrinsic buyers end up holding the security at time \( t = 1 \), and more intrinsic sellers end up being short the security at time \( t = 1 \). Similarly, the welfare is lower when fewer intrinsic buyers hold and fewer intrinsic sellers have sold the security at time \( t = 1 \). From this discussion, one can also consider the market welfare as the degree to which the intrinsic tendencies of the traders (prior to receiving any private information) are actualized.

In our welfare analysis, a central theme is the role of two potential transaction costs faced by the traders. One transaction cost is explicit in our model: any trader in the open market faces a transaction cost \( \delta \) set by the market maker. The second transaction cost is implicit: any trader in the dark pool faces an adverse selection cost (cf. Section 4.5); from Theorem 2, we know this cost is higher when the fill rate in the dark pool is lower. To a large extent, our welfare results are driven by the role of these transaction costs in shaping the trading decisions of intrinsic buyers and sellers. We compare two types of equilibria: those where a dark pool is present, and those where a dark pool is absent. By comparing the two transaction costs in the former with the open market transaction cost in the latter, we can get a sense of how welfare is affected by introduction of a dark pool. However, transaction costs, by themselves, do not explain the whole story. Intrinsic traders may go against their natural inclinations (for example, an intrinsic buyer may sell) given their private information. Higher transaction costs may inhibit such behavior and thus might improve welfare. The bottom line is that the overall impact of trading costs on welfare is not a priori obvious.

Our main result in this section is to use these insights to identify broad parameter regimes where the introduction of the dark pool either increases or decreases welfare. Largely, we argue that the welfare increase due to the introduction of dark pool occurs when its introduction facilitates relatively lesser informed intrinsic traders with moderately high idiosyncratic value to trade with lesser informed speculators. To support this thesis, we first consider in Section 5.2 a setting where there are no speculators (\( \mu = 0 \)). Regardless of the idiosyncratic value \( V \), we show that the introduction of a dark pool raises the transaction cost set by the market maker in the open market and decreases the welfare, relative to the competitive equilibrium without a dark pool. Thus, the presence of speculators is a precondition for welfare increase from introduction of a dark pool.

Consequently, in Section 5.3 we consider the asymptotic regime where the level of speculators \( \mu \) tends to infinity. In this regime, we find that the precise effect of introducing a dark pool depends on the magnitude of intrinsic traders’ idiosyncratic value. When the idiosyncratic value is too high, the traders are sufficiently motivated to realize the maximum possible welfare with only the open market, and hence the introduction of a dark pool can only lead to welfare reduction. On the other hand, when the idiosyncratic value is moderate, the informational concerns of the intrinsic traders precludes them from realizing the full welfare with only the open market. In such a setting, the introduction of a dark pool allows relatively lesser informed intrinsic traders to trade with the lesser informed speculators in the dark pool and realize the welfare gains.
5.1. Preliminaries: No dark pool

In this section, we define a reference competitive equilibrium NODP with no trade in the dark pool, against which we compare all equilibria with trade in the dark pool. The following result shows that such an equilibrium exists. The proof is deferred to the appendix.

**Lemma 3.** For any $\mu \geq 0$ and $V > 0$, there exists a unique competitive equilibrium, denoted by NODP, where the symmetric strategy profile involves trade only in the open market (i.e., the fill rate in the dark pool is zero). The transaction cost in the NODP equilibrium is given by

$$\delta_{\text{NODP}}(\mu, V) = 1 - \frac{\min\{V(\mu), V\}}{\sqrt{\mu + 1}},$$  

(21)

where $V(\mu) \triangleq \frac{2\sqrt{\mu + 1}}{\sqrt{\mu + 1} + 1}$. The welfare in the NODP equilibrium is given by

$$w_{\text{NODP}}(\mu, V) = \begin{cases} 
2V & \text{if } V \geq V(\mu); \\
V^2 \left(1 + \frac{1}{\sqrt{\mu + 1}}\right) & \text{if } V < V(\mu).
\end{cases}$$  

(22)

In the preceding lemma, for a given $\mu \geq 0$, the quantity $V(\mu)$ can be interpreted as the minimum idiosyncratic value for which the intrinsic traders’ exogenous motives for trade dominate their informational concerns. For $V \geq V(\mu)$, all intrinsic buyers buy and all intrinsic sellers sell at the open market in the NODP equilibrium. For $V < V(\mu)$, there exist intrinsic buyers (sellers) with sufficiently low (resp., high) signals, who trade against their intrinsic tendencies and sell (resp., buy) at the open market in the NODP equilibrium.

Note that, without the presence of the dark pool, the transaction cost in the open market would be set at $\delta_{\text{NODP}}(\mu, V)$ in a perfectly competitive market. In order to study how the presence of the dark pool affects the transaction costs in the open market, we will compare the transaction cost in any competitive equilibrium with trade in the dark pool to $\delta_{\text{NODP}}$. Similarly, the welfare implications of trade in the dark pool will be assessed by comparing the welfare of any competitive equilibrium with trade in the dark pool to the welfare of the NODP equilibrium.

5.2. A benchmark case: No speculators

In this section we consider a benchmark model, where there are no speculators ($\mu = 0$). We have two results. First, we show that the transaction cost in the presence of a dark pool is higher than $\delta_{\text{NODP}}$. We use this insight to show our main result: the introduction of a dark pool decreases welfare.

We start with the following theorem, which states that in any competitive equilibrium where there is trade in the dark pool, the transaction cost in the open market would be greater than or equal to $\delta_{\text{NODP}}$. The proof is deferred to the appendix.

**Theorem 3.** Suppose $\mu = 0$ and $V > 0$. For any transaction cost $\delta < \delta_{\text{NODP}}$, the market maker’s expected utility in any partial equilibrium $\lambda$ is negative.

Thus, we obtain the following corollary, stating that the presence of the dark pool increases the transaction cost set by a competitive market maker:

**Corollary 2.** Suppose $\mu = 0$ and $V > 0$. In any competitive equilibrium where the fill rate in the dark pool is positive, the transaction cost is greater than or equal to that in the NODP competitive equilibrium with no trade in the dark pool.

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10We suppress the dependence on the parameters $\mu$ and $V$ when the context is clear.
The intuition behind the preceding result is that the presence of a dark pool increases the adverse selection cost faced by a competitive market maker. In particular, as relatively uninformed traders move to trade in the dark pool, the population of traders in the open market becomes relatively better informed and collectively increases the adverse selection cost for the market maker. In order to compensate for this, a competitive market maker sets a higher transaction cost.

We next investigate how the presence of the dark pool affects welfare. Our main result in this section implies that the presence of the dark pool reduces the welfare of the market. The proof is available in Appendix B.

**Theorem 4.** Suppose \( \mu = 0 \) and \( V > 0 \). Suppose the transaction cost \( \delta \) satisfies \( \delta \geq \delta_{\text{NODP}}(0,V) \), and \( \lambda \) is a corresponding partial equilibrium. Then, the welfare under \( \lambda \) is less than or equal to that in the NODP competitive equilibrium with no trade in the dark pool.

Taken together with Theorem 3, we obtain the following result:

**Corollary 3.** Suppose \( \mu = 0 \) and \( V > 0 \). The welfare under any competitive equilibrium with a positive fill rate in the dark pool is less than or equal to the welfare of the NODP equilibrium with no trade in the dark pool.

We briefly describe the intuition behind the preceding result. Note that as emphasized above, welfare is improved if more intrinsic buyers hold the security (and symmetrically, more intrinsic sellers have sold the security). Now observe that the transaction cost in the open market rises with the introduction of a dark pool; this effect leads to fewer intrinsic buyers buying in the open market, a negative welfare effect. The only potential countervailing force is that some of these traders are enticed to buy in the dark pool instead: if that volume is sufficient, welfare may actually rise. However, in equilibrium, traders in the dark pool suffer an adverse selection cost; and we show that this cost is sufficient to actually cause a net reduction in the fraction of intrinsic buyers ultimately holding the security.

### 5.3. The market with speculators

The preceding section illustrates that when there are no speculators in the market, the introduction of a dark pool cannot increase the overall welfare of the market. This suggests that the presence of speculators is a necessary precondition to obtain welfare improvement upon introducing a dark pool. Consequently, in this section, we consider the asymptotic regime where the level of speculation is high, and further analyze the welfare consequences of the introduction of a dark pool. In this regime, we find that the precise effect on welfare of introducing a dark pool depends on magnitude of intrinsic traders’ idiosyncratic value. In particular, we obtain that when the level of speculation is sufficiently high and the idiosyncratic value is substantially high \( (V \geq 1.88) \), the introduction of a dark pool decreases the welfare of the market, as in the benchmark case. More interestingly however, we show that when idiosyncratic value \( V \) is moderately high \( (V \in [1, 1.879]) \) and the level of speculation is sufficiently high, there exists a competitive equilibrium involving trade in the dark pool that attains a higher welfare than that in the competitive equilibrium NODP with no trade in the dark pool.

Throughout this section, we assume that the idiosyncratic value \( V \) is greater than or equal to one. This assumption implies that the intrinsic traders’ exogenous motives for trade are comparable to or larger than their informational reasons for trade. (We study the case \( V < 1 \) numerically in Section 6.3) We begin by showing that, for all large enough \( \mu \), there exists a competitive equilibrium where both the intrinsic traders and the speculators trade in both the dark pool and the open market:
Theorem 5. Suppose $V \geq 1$. For all large enough $\mu \geq 0$, there exists a competitive equilibrium, denoted by BOTH, where both the intrinsic traders and the speculators trade in both the dark pool and the open market; the equilibrium fill rate $f_{\text{BOTH}}(\mu,V)$ and the equilibrium transaction cost $\delta_{\text{BOTH}}(\mu,V)$ in this equilibrium satisfy:\footnote{In what follows, given functions $f,g,q: \mathbb{R}_+ \rightarrow \mathbb{R}_+$, we say that $f = g + o(q)$ if $\limsup_{\mu \rightarrow \infty} |f(\mu) - g(\mu)|/q(\mu) = 0$, i.e., if the difference between $f$ and $g$ converges to 0 at a faster rate than $q$. Similarly, we say that $f = g + \Theta(q)$ if $0 < \liminf_{\mu \rightarrow \infty} |f(\mu) - g(\mu)|/q(\mu) \leq \limsup_{\mu \rightarrow \infty} |f(\mu) - g(\mu)|/q(\mu) < \infty$, i.e., if the difference between $f$ and $g$ converges to 0 at the same rate as $q$.}

$$f_{\text{BOTH}}(\mu,V) = \frac{C_1}{\sqrt{\mu}} + o\left(\frac{1}{\sqrt{\mu}}\right), \quad \delta_{\text{BOTH}}(\mu,V) = 1 - \frac{C_2}{\sqrt{\mu}} + o\left(\frac{1}{\sqrt{\mu}}\right),$$

where $C_1, C_2 > 0$ are (known) constants that depend on $\mu$.

The following result compares the transaction costs in the BOTH equilibrium with that in the NODP equilibrium. The proof is deferred to the appendix.

Theorem 6. For all $V \geq 1$, we have

$$\lim_{\mu \rightarrow \infty} \frac{1 - \delta_{\text{BOTH}}(\mu,V)}{1 - \delta_{\text{NODP}}(\mu,V)} = \frac{1}{\min\{2,V\}} \frac{\sqrt{1+2V+4V^2}}{1+V}.$$  

In particular, for all $V \geq 1.428$, the preceding limit is strictly less than 1, whereas for all $V \in [1,1.427]$, the limit is strictly greater than 1.

The preceding theorem states that if the idiosyncratic value $V$ is significantly greater than one, implying that the exogenous hedging motive of the intrinsic traders is substantially high, then for high levels of speculation, the transaction cost in the open market in the BOTH equilibrium is larger than that in the NODP equilibrium. Thus, under this setting, the presence of a dark pool increases the transaction costs in the open market in equilibrium. On the other hand, when the idiosyncratic value $V$ is moderately high, the presence of a dark pool decreases the transaction costs in the open market for large values of $\mu$.

Finally, we compare the resulting welfare $w_{\text{BOTH}}(\mu,V)$ in the BOTH equilibrium with $w_{\text{NODP}}(\mu,V)$, the welfare in the NODP equilibrium without trade in the dark pool. The proof is deferred to the appendix.

Theorem 7. For all $V \geq 1$, we have

$$\lim_{\mu \rightarrow \infty} \frac{w_{\text{BOTH}}(\mu,V)}{w_{\text{NODP}}(\mu,V)} = \frac{1}{\min\{2,V\}} \left(2 - \frac{1}{(1+V)^2}\right).$$

In particular, for values of $V \geq 1.88$, the preceding limit is strictly less than 1, whereas for all $V \in [1,1.879]$, the limit is strictly greater than 1.

Thus, the preceding theorem states that when the level of speculation is high, the welfare in the BOTH equilibrium is strictly less than that in the NODP equilibrium for significantly high idiosyncratic value. Thus, as in the benchmark case, under this setting the presence of dark pool decreases the welfare of the market. However, more interestingly, we observe that when the level of speculation is high and the idiosyncratic value is moderately high, implying that the intrinsic traders’ exogenous hedging motives do not substantially dominate their informational reasons for trade, the presence of a dark pool improves the welfare of the market.
To see the intuition behind these results, first suppose $V \geq 2$, and consider the traders’ strategies in the limit where $\mu = \infty$. In the NODP equilibrium, all intrinsic buyers buy the asset and all intrinsic sellers sell the asset in the open market. On the other hand, in the BOTH equilibrium, all intrinsic buyers with signal $s \leq 1/(1 + V)$ enter buy orders in the dark pool, whereas those with $s > 1/(1 + V)$ buy in the open market. (The strategy for intrinsic sellers is symmetric.) From this characterization of the equilibrium strategies, we have the following two consequences:

1. First, since all traders realize their intrinsic tendencies in the NODP equilibrium, the welfare therein is the maximum possible, and in particular, larger than that in the BOTH equilibrium. Informally, when the traders’ exogenous motives to trade is substantially high, the presence of a dark pool draws traders away from the open market, who then forgo guaranteed execution in the open market and incur adverse selection in the dark pool, contributing to welfare loss.

2. Second, since all intrinsic traders trade in the open market in NODP equilibrium, the market maker incurs adverse selection only from the speculators, and not from the intrinsic traders. On the other hand, in the BOTH equilibrium, the market maker incurs adverse selection from both the intrinsic traders as well as the speculators. To compensate for this additional adverse selection cost, the market maker sets a higher transaction cost in the BOTH equilibrium.

Next, suppose $V < 2$ and again consider the traders’ strategies in the limit where $\mu = \infty$. In the NODP equilibrium, all intrinsic buyers with signal $s > 1 − V/2$ buy in the open market, whereas those with $s \leq 1 − V/2$ choose not to trade. On the other hand, in the BOTH equilibrium, all intrinsic buyers with signals $s > 1/(1 + V)$ buy in the open market, whereas those with $s \leq 1/(1 + V)$ enter buy order in the dark pool. (The strategies for intrinsic sellers in either equilibrium are symmetric.) Note that, since the (buy) fill rate in the BOTH equilibrium is essentially zero when $\sigma = 1$, essentially no intrinsic buyer with signal $s < 1/(1 + V)$ ends up holding the asset when the asset value increases. Since $1 − V/2 < 1/(1 + V)$ for $V < 2$, this implies that, when $\sigma = 1$, fewer intrinsic buyers hold the asset in the BOTH equilibrium as compared to the NODP equilibrium. However, when $\sigma = −1$, all buy orders in the dark pool get filled. Thus, when the asset value decreases, all intrinsic buyers end up buying the asset in the BOTH equilibrium, either in the open market or from speculators in the dark pool. Thus, when $\sigma = −1$, more intrinsic buyers buy the asset in the BOTH equilibrium than in the NODP equilibrium. (The situation is symmetric for intrinsic sellers.) The interaction between these two aspects directly impacts whether the NODP equilibrium has higher welfare that the BOTH equilibrium or vice versa. For small enough values of $V \geq 1$, namely $V \in [1, 1.879)$, the latter effect, namely all intrinsic buyers being able to realize their intrinsic tendencies, dominates, and the welfare in the BOTH equilibrium is higher.

These results further lend support to our thesis that the welfare improvement on the introduction of a dark pool arises mainly from having relatively uninformed intrinsic traders with moderately high idiosyncratic value trade with uninformed speculators. In general, we find the same result holds numerically for a wide range of the problem parameter values $(\mu, V)$ — our numerical investigation is presented in the following section.

6. Computational experiments

In this section, we augment our analytical results with supporting numerical evidence from extensive equilibrium computations over a broad parameter regime. We begin by describing the numerical approach, and provide illustrations of the threshold strategies of the traders. We then provide two sets of numerical results for the comparative statics of competitive equilibria. The first set studies the benchmark case with no speculators, and, in particular, provides comparative...
statics with respect to the intrinsic value $V$; as our theoretical results in Section 5.2 demonstrate, in this regime introduction of the dark pool causes welfare to fall. The second set studies the effect of increasing the mass of speculators $\mu$, when traders have a moderately high intrinsic value for trade; as the results in Section 5.3 suggest, in this regime, welfare increases for sufficiently large values of $\mu$.

6.1. Numerical approach

Recall that for any fixed values of the mass of speculators $\mu$, the intrinsic value $V$, and the transaction cost $\delta$, a partial equilibrium with symmetric strategy profile is a strategy profile in the class $\mathcal{H}(\delta)$ that is the fixed point of the best response map $\Lambda[\mu, V, \delta; \cdot]$. Here, we make explicit the dependence of $\Lambda[\cdot]$ on the model parameters. Since the class of strategy profiles $\mathcal{H}(\delta)$ is parameterized by a single (buy) fill rate parameter $f \in [0, 1]$, we can search numerically for all partial equilibria with symmetric strategy profile by searching over the possible values for the fill rate in the interval $[0, 1]$. In particular, we iterate over a discrete set $\mathcal{F} \subseteq [0, 1]$ of values for the fill rate $f$, and compute the resulting fill rate parameter of the best response strategy profile $\Lambda[\lambda^f, \delta]$. We store all values of $f$ for which the absolute value of the difference between $f$ and the fill rate parameter of $\Lambda[\lambda^f, \delta]$ is below a small threshold $\epsilon_1$. This set of values of $f$ then identifies approximately all partial equilibria with symmetric strategy profile for any transaction cost $\delta \in [0, 1]$ and for given values of the model parameters $\mu$ and $V$.

We perform another numerical search to identify those values of $\delta \in [0, 1]$ that lead to zero expected utility for the market maker in at least one of the corresponding partial equilibria. More precisely, we iterate over a discrete set $\mathcal{D} \subseteq [0, 1]$ of values for the transaction cost $\delta \in [0, 1]$, and for each of the corresponding partial equilibria computed earlier, we compute the market maker’s expected utility. We store those values of $(\delta, f)$ of the partial equilibria for which the absolute value of the market maker’s expected utility is below a small threshold $\epsilon_2$. The value of the transaction cost $\delta$ along with corresponding partial equilibrium fill rate $f$ together then identify a competitive equilibrium for the given values of the model parameters, up to a small numerical tolerance.\(^{12}\)

Note that in the numerical results that follow, for a given set of model parameters, there are multiple equilibria. We distinguish between the equilibria as follows: the NODP equilibrium, which involves no trade in the dark pool, is labeled “NODP”. The equilibria with trade in the dark pool vary along continuous curves as model parameters are changed. We label these equilibria as belonging to one of two branches, in order to clarify the relationship between the sets of equilibria in different subfigures.

6.2. The benchmark case: No speculators

In Figure 2, we consider, for different idiosyncratic values $V \in (0, 1]$, the benchmark case of no speculation, i.e., $\mu = 0$. In Figure 2(b) we plot the adverse selection cost faced by an uninformed trader ($s = 0.5$) in the dark pool in the different competitive equilibria with trade in the dark pool, for different values of the idiosyncratic value $V$. In Figures 2(a) and 2(c), we plot respectively the transaction cost in the open market and the welfare, in different competitive equilibria, for different values of the idiosyncratic value $V$. Note that for values of $V$ below approximately 0.42, there are no competitive equilibria with trade in the dark pool. Furthermore, given any equilibrium with no trade in the open market, as the transaction cost increases, equilibrium is maintained. Hence, we have the shaded region in the upper right corner of Figure 2(a).

\(^{12}\)We make the following specific choices for the thresholds and the discrete sets for our numerical results: $\mathcal{F} = \{k/10000 : k = 0, 1, \cdots, 10000\}$, $\mathcal{D} = \{k/1000 : k = 0, 1, \cdots, 1000\}$, $\epsilon_1 = 10^{-3}$, and $\epsilon_2 = 4 \cdot 10^{-4}$.
First and foremost, we see from these figures that, for a given $V$, the transaction cost in any competitive equilibria with trade in the dark pool is higher than that in the NODP equilibrium. Similarly, for a given $V$, the welfare in an equilibrium with trade in the dark pool is lower than that in the NODP equilibrium. This is consistent with our analytical results in Section 5.2.

Second, from Figure 2(b), we see that the adverse selection cost in the dark pool decreases as the idiosyncratic value $V$ increases. Furthermore, we see that for values of $V$ greater than approximately 0.6, there are two sets of competitive equilibria with trade in the dark pool. In the lower set of equilibria (the first branch), the adverse selection cost decreases to zero as $V$ increases to one. In the upper set of equilibria (the second branch), although the adverse selection cost decreases to a positive value approximately equal to 0.33, Figure 2(d) reveals that the volume of orders in the dark pool converges to zero.

These figures reveal the intricate connection between transaction costs, adverse selection costs, and welfare. In particular, because the dark pool leads the transaction cost in the open market to increase, and the adverse selection cost is significant, welfare falls. Notice that the relative decrease in welfare is lower as $V \to 1$; this results because both the transaction cost and the welfare in the competitive equilibria with trade in the dark pool approach that in the NODP equilibrium. As intrinsic traders become more highly motivated to trade, the welfare losses incurred by introduction of the dark pool are naturally mitigated.

6.3. The market with speculators: The case of moderately high idiosyncratic value

Next, we turn our attention to the market with speculators. We first consider, in Figure 3, the case of moderately high idiosyncratic value, with $V = 0.9$. In Figure 3(b), we plot the adverse selection cost faced by an uninformed trader ($s = 0.5$) in the dark pool in different competitive equilibria with trade in the dark pool, for different values of the mass of speculators $\mu$. In Figures 3(a) and 3(c), we plot respectively the transaction cost in the open market and the welfare, in different competitive equilibria, for different values of the mass of speculators $\mu$. As before, given any equilibrium with no trade in the open market, as the transaction cost increases, equilibrium is maintained. Hence, we have the shaded region in the upper left corner of Figure 3(a).

From Figure 3(c), we see that for $\mu$ greater than approximately 3, there exists a competitive equilibrium with trade in the dark pool (in the first branch) such that the welfare is higher as compared to the NODP equilibrium. This observation expands on the asymptotic analytical results in Section 5.3; in this case, $\mu$ is large but finite, and $V$ is moderately high, but strictly less than one, and the welfare increases on introduction of the dark pool.

If we try to understand the roots of this effect, we are led to study the transaction cost in the open market (cf. Figure 3(a)) and the adverse selection cost on introduction of the dark pool (cf. Figure 3(b)). We have two main observations. First, on the first branch, the transaction cost in the open market is higher with the presence of the dark pool when $\mu$ is low, but eventually behaves similarly to (and is slightly lower than) the transaction cost in the open market without the dark pool. Second, the adverse selection cost is significant throughout the range of $\mu$ we consider. However, because $V$ is moderately high, intrinsic buyers and sellers are still motivated to trade. Since the dark pool does not materially impact the transaction costs in the open market, the presence of the dark pool merely acts as another venue for intrinsic buyers and sellers to trade, resulting in welfare gains. This matches the analytical findings in Section 5.3.
(a) Transaction cost.

(b) Adverse selection in the dark pool.

(c) Welfare.

(d) Volume of orders in dark pool.

(e) Total volume of orders.

(f) Fraction of order volume in dark pool.

**Figure 2:** Competitive equilibria in the benchmark case with no speculators ($\mu = 0$).
Figure 3: Competitive equilibria with speculators, for moderately high idiosyncratic value \((V = 0.9)\).
7. Conclusion

Our main goal in this paper is to analyze the welfare implications of operating a dark pool alongside traditional lit markets. We consider a stylized model of a competitive market where traders have heterogeneous fine-grained private information about the short-term future price of the asset.

The main conclusion of the paper is that the overall effect of a dark pool on the market welfare is driven by two competing forces. On one hand, in equilibrium, the orders in the dark pool face an implicit transaction cost in the form of adverse selection. We show that this adverse selection occurs despite the fact that highly informed traders trade in the open market, whereas the dark pool is populated with orders from relatively moderately informed traders. This implicit adverse selection cost in the dark pool induces a detrimental effect on the overall welfare of the market. On the other hand, the dark pool also provides an additional venue for trade. In particular, the dark pool facilitates relatively lesser informed intrinsic traders to trade with lesser informed speculators. When the level of speculation is high, this creates additional opportunities for trade for those intrinsic traders submitting orders against the asset price movement. This additional liquidity induces a positive effect on the overall welfare. The interplay between the detrimental adverse selection effect and the positive liquidity effect determines how the dark pool affects the overall welfare of the market. In particular, our analysis shows that when there is no speculation in the market, the introduction of a dark pool decreases the market welfare. Similarly, when the level of speculation is high, but the intrinsic traders have substantially high external motives for trade, the dark pool does not increase the market welfare. It is in presence of high levels of speculation and intrinsic traders with moderately high idiosyncratic values that the introduction of a dark pool leads to increase in the overall welfare of the market.

We show that the welfare improvement in the presence of a dark pool is accompanied with a decrease in the transaction cost in the open market. This finding is important because it is in contrast to empirical findings that suggest that the introduction of a dark pool typically raises transaction costs in the lit market (e.g., Comerton-Forde and Putniņš 2015; Degryse et al. 2014; Foley et al. 2012). Our model suggests that in parameter regimes where transaction costs in the open market increase on introduction of a dark pool, welfare typically decreases. As a result, our paper finds that the regulators have a legitimate concern about the potentially negative welfare implications of the introduction of dark pools alongside traditional lit markets.

Our modeling assumptions are primarily geared towards understanding the impact of asymmetric information on the gains from trade. There are many other aspects to dark pool trading that are interesting (e.g., questions of price efficiency, or information transmission across venues, etc.). Addressing these aspects in the presence of asymmetric information constitutes an important area for future research.

References


### A. Proofs

**Proof of Lemma 1** Suppose the strategy profile $\lambda$ is symmetric. Then, it follows directly from the definition that, for any $v \in \{0, \pm V\}$ and any $s \in [0, 1]$, we have $s \in \Theta^\lambda_{v, (DP,B)}$ if and only if $1 - s \in \Theta^\lambda_{-v, (DP,S)}$. Thus, we obtain for all $v \in \{0, \pm V\}$ and $\sigma \in \{\pm 1\}$,

$$F_\sigma(\Theta^\lambda_{v, (DP,B)}) = F_\sigma(1 - \Theta^\lambda_{-v, (DP,S)}) = F_{-\sigma}(\Theta^\lambda_{-v, (DP,S)}).$$

Here, for any set $A$, we have defined $1 - A \triangleq \{1 - x : x \in A\}$. The second line follows from the symmetry of the signal structure. In particular, we have $f_1(s) = f_{-1}(1 - s)$ for all $s \in [0, 1]$, where $f_\sigma$ is the density of $F_\sigma$. This implies that for any set $A$, $F_\sigma(A) = F_{-\sigma}(1 - A)$.

Thus, from (9), we obtain

$$m^\lambda_B(\sigma) = \sum_{v \in \{\pm V\}} F_\sigma(\Theta^\lambda_{v, (DP,B)}) + \mu F_\sigma(\Theta^\lambda_{0, (DP,B)}) = \sum_{v \in \{\pm V\}} F_\sigma(1 - \Theta^\lambda_{-v, (DP,S)}) + \mu F_\sigma(1 - \Theta^\lambda_{0, (DP,S)}) = \sum_{v \in \{\pm V\}} F_{-\sigma}(\Theta^\lambda_{-v, (DP,S)}) + \mu F_{-\sigma}(\Theta^\lambda_{0, (DP,S)}) = m^\lambda_S(-\sigma).$$

The last equality follows from (10). Thus, when the strategy profile is symmetric and $m^\lambda_B(\sigma)$ is positive for all $\sigma$, the fill rates, as defined in (11), satisfy

$$\phi^\lambda_B(\sigma) = \phi^\lambda_S(-\sigma) = \min \left(1, \frac{m^\lambda_S(\sigma)}{m^\lambda_B(\sigma)}\right), \quad \text{for} \quad \sigma \in \{-1, 1\}.$$  

From this, (12) and (13), it is straightforward to verify that

$$u^\lambda_{OM,B}(v, s) = 2s - 1 + v - \delta$$
\[ u^\lambda_{(OM,S)}(v, s) = 2(1 - s) - (-v) - \delta \]
\[ u_{(OM,B)}^\lambda(v, s) = u_{(OM,S)}^\lambda(v, s) = (1 - 2s)(-v) - \delta \]

Furthermore, we have \( u_N(v, s) = 0 = u_N(-v, 1 - s) \). Thus, while computing the best response strategies, the decision problem faced by a trader with idiosyncratic value \( v \) at signal \( s \) is equivalent to that of a trader with idiosyncratic value \(-v\) at signal \( 1 - s \), with the qualification that whenever the former buys, the latter sells (and vice versa). This suffices to conclude that the best response strategy profile \( \Lambda^\lambda \) is symmetric. ■

**Proof of Theorem 1.** Consider a symmetric strategy profile \( \lambda \) with trade in the dark pool such that \( \phi_B^\lambda(-1) = \phi_S^\lambda(1) = f < 1 \) and \( \phi_B^\lambda(1) = \phi_S^\lambda(-1) = 1 \). The traders’ expected utility can be simplified as

\[
\begin{align*}
&u_{OM,B}^\lambda(v, s) = 2s - 1 + v - \delta, \\
&u_{OM,S}^\lambda(v, s) = 1 - 2s - v - \delta, \\
&u_{DP,B}^\lambda(v, s) = s(1 + v) + (1 - s)(-1 + v) \\
&\quad \quad \quad = s(1 + v + f(1 - v)) - f(1 - v), \\
&u_{DP,S}^\lambda(v, s) = sf(-1 - v) + (1 - s)(1 - v) \\
&\quad \quad \quad = 1 - v - s(f(1 + v) + 1 - v), \\
&u_N^\lambda(v, s) = 0.
\end{align*}
\]

From these expressions, it is straightforward to conclude that if \( v \in (0, 1] \), then

\[
\Lambda^\lambda_v(s) = \begin{cases} 
(DP, B) & \text{for } s \geq \frac{1 - v}{2}; \\
(DP, S) & \text{for } s < \frac{1 - v}{2},
\end{cases}
\]

and if \( v > 1 \), then

\[
\Lambda^\lambda_v(s) = \begin{cases} 
(DP, B) & \text{for } s \geq \left(1 - \frac{\delta}{(v - 1)(1 - f)}\right)^+; \\
(OM, B) & \text{for } s < \left(1 - \frac{\delta}{(v - 1)(1 - f)}\right)^+,
\end{cases}
\]

Define \( \bar{x} \) and \( \bar{y} \) as follows:

\[
\bar{x} = \begin{cases} 
\frac{1 - V}{2} & \text{if } V \in (0, 1]; \\
\left(1 - \frac{\delta}{(V - 1)(1 - f)}\right)^+ & \text{if } V > 1,
\end{cases}
\]
\[
\bar{y} = \begin{cases} 
\frac{1 - V}{2} & \text{if } V \in (0, 1]; \\
0 & \text{if } V > 1,
\end{cases}
\]

Note that under the strategy profile \( \gamma \triangleq \Lambda^\lambda \), intrinsic buyers with signals \( s \geq \bar{x} \) choose to submit a buy order in the dark pool, and intrinsic buyers with signals \( s < \bar{y} \) choose to submit a
sell order in the dark pool. By symmetry of the strategy profile \( \gamma \), this implies that, the mass of buy orders in the dark pool when \( \sigma = 1 \) satisfies

\[
m^\gamma_B(1) = F_1 [\bar{x}, 1] + F_1 [1 - \bar{y}, 1] + \mu F_1 \left[ \frac{1}{2}, 1 \right]
\]

\[
= F_{-1} [0, 1 - \bar{x}] + F_{-1} [0, \bar{y}] + \mu F_{-1} \left[ 0, \frac{1}{2} \right]
\]

\[
\geq F_1 [0, 1 - \bar{x}] + F_1 [0, \bar{y}] + \mu F_1 \left[ 0, \frac{1}{2} \right]
\]

\[
= m^\gamma_S(1).
\]

Here, we have used in the second line the fact that \( F_{-\sigma}(A) = F_{-\sigma}(1 - A) \) for any set \( A \). The inequality follows from the observation that for any \( x \), we have \( F_1 [0, x] = x^2 \leq 1 - (1 - x)^2 = F_{-1} [0, x] \).

Thus, we obtain \( m^\gamma_B(1) \geq m^\gamma_S(1) \). If \( m^\gamma_B(1) = 0 \), then, under \( \gamma \), there is no trade in the dark pool, implying \( \gamma \neq \lambda \). On the other hand, if \( m^\gamma_B(1) > 0 \), we obtain \( \phi_S^\gamma(1) = \min\{1, \frac{m^\gamma_B(1)}{m^\gamma_S(1)}\} = 1 \). Since \( \phi_S^\gamma(1) < 1 \), this shows that \( \lambda \neq \gamma \). Thus, in both cases, we obtain \( \lambda \neq \gamma = A^\lambda \), and hence \( \lambda \) is not a BNE.

This implies that for all values of \( V > 0 \), \( \delta \in [0, 1] \) and \( \mu \geq 0 \), any partial equilibrium \((\lambda, \delta)\) with symmetric strategy profile \( \lambda \) involving trade in the dark pool must satisfy \( 0 < \phi_S^\lambda(1) = \phi_S^\lambda(1) \leq 1 \) and \( \phi_B^\lambda(-1) = \phi_B^\lambda(1) = 1 \).

**Proof of Lemma 2** Let \( f \in [0, 1] \) and \( v \geq 0 \). It is straightforward to verify that \( \Theta_{v,(OM,B)} \cap \Theta_{v,(DP,B)} = \emptyset \), \( \Theta_{v,(OM,B)} \cap \Theta_{v,N} = \emptyset \), and \( \Theta_{v,(DP,B)} \cap \Theta_{v,N} = \emptyset \). Similarly, \( \Theta_{v,(OM,S)} \cap \Theta_{v,(DP,S)} = \emptyset \), \( \Theta_{v,(OM,S)} \cap \Theta_{v,N} = \emptyset \), and \( \Theta_{v,(DP,S)} \cap \Theta_{v,N} = \emptyset \).

Now, suppose there exists an \( s \in \Theta_{v,(OM,B)} \cap \Theta_{v,(OM,S)} \). By definition, this implies that \( v \in [0,1] \) and \( (1 - v - \delta)/2 > s > (1 - v + \delta)/2 \), which contradicts \( \delta \geq 0 \). Also, if there exists an \( s \in \Theta_{v,(DP,B)} \cap \Theta_{v,(DP,S)} \), we must have \( v \in [0, 1] \) and \( f(1 - v)/(f(1 - v) + 1 + v) > s \) which contradicts \( f \leq 1 \). Similarly, for \( s \in \Theta_{v,(OM,B)} \cap \Theta_{v,(DP,S)} \), we have \( v \in [0, 1] \) and \( f(1 - v)/(f(1 - v) + 1 + v) > s \) which contradicts \( \delta \geq 1 \). A similar argument shows that \( \Theta_{v,(DP,B)} \cap \Theta_{v,(OM,S)} = \emptyset \). This shows that the intervals are mutually exclusive for any \( v \geq 0 \). Since the intervals are defined symmetrically for \( v < 0 \), the result holds for all values of \( v \in \mathbb{R} \).

Next, if \( v > 1 \), it is straightforward to show that the union of the intervals is \([0, 1]\). Hence, suppose \( v \in [0, 1] \). Observe that \( (1 - v + \delta)/2 > (1 - v)/(1 - v + f(1 + v)) \) if and only if \( \delta \geq (1 - f)/(1 + v) \) if and only if \( \delta \geq (1 - f)/(1 + v) \). Using this and the expressions from \([19]\), it is straightforward to show that

\[
\Theta_{v,(OM,B)} \cup \Theta_{v,(DP,B)} = \left( \min \left\{ \frac{1 - v + \delta}{2}, \frac{1 - v}{1 - v + f(1 + v)} \right\}, 1 \right).
\]

By a similar argument using the fact that \( (1 - v - \delta)/2 > f(1 - v)/(f(1 - v) + 1 + v) \) if and only if \( 1 - \delta/(1 - v)(1 - f) \geq \max\{(1 - v)/(1 - v + f(1 + v)), (1 - v + \delta)/2\} \), we obtain

\[
\Theta_{v,(OM,S)} \cup \Theta_{v,(DP,S)} = \left[ 0, \max \left\{ \frac{1 - v - \delta}{2}, f(1 - v)/(f(1 - v) + 1 + v) \right\} \right).
\]

This implies that \( \Theta_{v,(OM,B)} \cup \Theta_{v,(DP,B)} \cup \Theta_{v,(OM,S)} \cup \Theta_{v,(DP,S)} = [0, 1]\) for all \( v \in [0, 1] \). Together, this implies that the intervals are collectively exhaustive for all values of \( v \geq 0 \). Again,
since the intervals are defined symmetrically for \( v < 0 \), the result extends to all values of \( v \in \mathbb{R} \). This completes the proof of part \((i)\).

Next, suppose \( \lambda \) is a symmetric strategy profile with \( \phi_{\lambda}^h(1) = f \in (0,1] \), and \( \phi_{\lambda}^s(1) = 1 \). Given this, it is straightforward to verify that a trader’s utility functions for different actions are linear in \( s \). This implies that the best response strategy profile \( \Lambda[\lambda] \) has a simple threshold structure, where the thresholds correspond to those signal values where two (or possibly more) actions yield the same expected utility. Finally, given the fill rate \( f = \phi_{\lambda}^h(1) \), one can verify through straightforward calculations that these thresholds correspond exactly to those of \( \lambda^{I,\delta} \in H(\delta) \).

Finally, suppose the symmetric strategy profile \( \lambda \) has fill rates \( \phi_{\lambda}^h(1) = \phi_{\lambda}^s(1) = 0 \). Since this can arise only if there is a zero mass of buy (or sell) orders in the dark pool, the best response strategy \( \Lambda[\lambda] \) would never involve submitting an order to the dark pool, and would only involve orders in the open market. Given this and using the fact that a trader’s utility for different actions are linear in \( s \), again we obtain through straightforward computation that the best response strategy profile \( \Lambda[\lambda] \) has a threshold structure, with thresholds corresponding exactly to that of \( \lambda^{0,\delta} \in H(\delta) \). This completes the proof of part \((ii)\).

\[ \textbf{Proof of Theorem 2} \]

Consider a partial equilibrium \( \lambda \) involving symmetric strategies, with trade in the dark pool. A trader with signal \( s \in [0,1] \) submitting a buy order in the dark pool, has the following belief about the value of the asset upon her order being filled:

\[
P(\sigma = 1|s, \text{buy fill}) = \frac{P(\text{buy fill}|s, \sigma = 1)P(\sigma = 1|s)}{P(\text{buy fill}|s, \sigma = 1)P(\sigma = 1|s) + P(\text{buy fill}|s, \sigma = -1)P(\sigma = -1|s)}
\]

\[
= \frac{sP(\text{buy fill}|\sigma = 1)}{sP(\text{buy fill}|\sigma = 1) + (1-s)P(\text{buy fill}|\sigma = -1)}.
\]

Here, the first equation follows from Bayes’ rule, the second equation follows from the fact that the signal structure satisfies \( P(\sigma = 1|s) = s \), and that conditional on \( \sigma \), the event \( I\{\text{buy fill} \} \) is independent of the signal \( s \), as the trade in the dark pool is through uniform matching.

Now, by definition of the fill rates, we have \( P(\text{buy fill}|\sigma) = \phi_{\lambda}^s(\sigma) \). This implies that,

\[
P(\sigma = 1|s, \text{buy fill}) = \frac{s\phi_{\lambda}^h(1)}{s\phi_{\lambda}^h(1) + (1-s)\phi_{\lambda}^s(-1)}.
\]

Since \( E[\sigma|s] = 2s - 1 \), and \( E[\sigma|s, \text{buy fill}] = 2P(\sigma = 1|s, \text{buy fill}) - 1 \), we obtain

\[
E[\sigma|s] - E[\sigma|s, \text{buy fill}] = 2s - \frac{s\phi_{\lambda}^h(1)}{s\phi_{\lambda}^h(1) + (1-s)\phi_{\lambda}^s(-1)} = \frac{2s(1-s)}{s\phi_{\lambda}^h(1) + (1-s)\phi_{\lambda}^s(-1)} \left( \phi_{\lambda}^h(-1) - \phi_{\lambda}^s(1) \right).
\]

From Theorem \((i)\), we know that in the partial equilibrium \( \lambda \), the fill rates satisfy \( \phi_{\lambda}^h(-1) = 1 \) and \( \phi_{\lambda}^s(1) \leq 1 \). Thus, we obtain that

\[
E[\sigma|s] - E[\sigma|s, \text{buy fill}] \geq 0, \quad \text{for all } s \in [0,1].
\]

Thus all traders submitting a buy order in the dark pool suffer from adverse selection. The proof for a trader submitting a sell order follows symmetrically.

Finally, we have

\[
\text{Adv}(\lambda) \triangleq E[\sigma|s = \frac{1}{2}, \text{buy fill}] - E[\sigma|s = \frac{1}{2}] = \frac{1 - \phi_{\lambda}^h(1)}{1 + \phi_{\lambda}^s(1)}.
\]
From this expression, using Theorem 14, it is straightforward to show that
\[
\text{Adv}(\lambda) = \mathbb{E}\left[\sigma \mid s = \frac{1}{2}, \text{sell fill}\right] - \mathbb{E}\left[\sigma \mid s = \frac{1}{2}\right] \geq 0.
\]

Furthermore, again from the expression, we obtain that \(\text{Adv}(\lambda)\) is a strictly decreasing function of the buy (sell) fill rate \(\phi^b_\delta(1) = \phi^b_\delta(-1)\).

**Proof of Lemma 3.** For any \(\delta \in [0, 1]\) and \(V > 0\), consider \(\lambda^{0,\delta}\), the partial equilibrium involving trade only in the open market. Using (19), we obtain the following structure for \(\lambda^{0,\delta}\):

\[
\lambda^{0,\delta}_0(s) = \begin{cases} 
(OM, B), & \text{if } s > \frac{1+\delta}{2}; \\
(OM, S), & \text{if } s < \frac{1-\delta}{2}; \\
N, & \text{otherwise},
\end{cases}
\]

and \(\lambda^{0,\delta}_V(s)\) is defined symmetrically. Here \(x(\delta) = \max\{\frac{1-V+\delta}{2}, 0\}\) and \(y(\delta) = \max\{\frac{1-V-\delta}{2}, 0\}\). Thus, the market maker’s payoff in this partial equilibrium is given by,

\[
\begin{align*}
u(\delta, \lambda^{0,\delta}) &= \delta \left(1 - F_1(x(\delta)) + F_1(y(\delta)) + 1 - F_1(1-y(\delta)) + F_1(1-x(\delta))\right) \\
&\quad + \mu \delta \left\{1 - F_1\left(\frac{1+\delta}{2}\right) + F_1\left(\frac{1-\delta}{2}\right)\right\} \\
&\quad - \left\{1 - F_1(x(\delta)) - F_1(y(\delta))\right\} - \left\{1 - F_1(1-y(\delta)) - F_1(1-x(\delta))\right\} \\
&\quad - \mu \left\{1 - F_1\left(\frac{1+\delta}{2}\right) - F_1\left(\frac{1-\delta}{2}\right)\right\}.
\end{align*}
\]

Here, the first two lines represent the market maker’s revenue from trade, and the last two lines represent the cost due to adverse selection. These expressions for the revenue and the adverse selection cost follow from the threshold structure of \(\lambda^{0,\delta}\), and from the fact that \(F_1(x) = 1 - F_{-1}(1-x)\).

Using the fact that \(F_1(x) = x^2\) for \(x \in [0, 1]\), we obtain,

\[
u(\delta, \lambda^{0,\delta}) = \begin{cases} 
\frac{1}{2} (V^2 - (1 + \mu)(1 - \delta)^2), & \text{if } \delta \geq |1-V|; \\
\frac{1}{2} (2V^2 - 2 + \mu)(1 - \delta)^2), & \text{if } V < 1, \text{ and } \delta < 1 - V; \\
\frac{1}{2} (4\delta - \mu(1 - \delta)^2), & \text{if } V > 1, \text{ and } \delta < V - 1.
\end{cases}
\]

(23)

Note from the expressions that \(\nu(\delta, \lambda^{0,\delta})\) is continuous and strictly increasing in \(\delta\). Observe that

\[
u(0, \lambda^{0,0}) = -\frac{\mu}{2} + \min\{V^2 - 1, 0\} \leq 0,
\]

\[
u(1, \lambda^{0,1}) = \min\{\frac{V^2}{2}, 2\} > 0.
\]

This implies that there is a unique \(\delta = \delta_{\text{NODP}}(\mu, V) \in [0, 1]\) such that \(\nu(\delta, \lambda^{0,\delta}) = 0\). Thus, there exists a unique competitive equilibrium \((\delta, \lambda^{0,\delta})\), where \(\delta = \delta_{\text{NODP}}(\mu, V)\), with trade only in the open market. We denote this competitive equilibrium by \(\text{NODP}\). From a straightforward calculation, we obtain

\[
\delta_{\text{NODP}}(\mu, V) = \begin{cases} 
1 - \frac{2}{\sqrt{\mu+1}}, & \text{if } V \geq V(\mu); \\
\frac{V}{\sqrt{\mu+1}}, & \text{if } V < V(\mu),
\end{cases}
\]

where \(V(\mu) \triangleq \frac{2\sqrt{\mu+1}}{\sqrt{\mu+1}+1}\).
Finally, for the partial equilibrium $(\delta, \lambda^{0,\delta})$, the welfare $w(\delta, \lambda^{0,\delta})$ is given by

$$w(\delta, \lambda^{0,\delta}) = V(1 - F_1(x(\delta)) - F_1(y(\delta)) - (1 - F_1(1 - y(\delta))) + F_1(1 - x(\delta)))$$

$$= 2V(1 - x(\delta) - y(\delta)).$$

For $\delta = \delta_{NODP}(\mu, V)$, on using the expressions for $x(\delta)$ and $y(\delta)$, this simplifies to provide the following expression for the welfare $w_{NODP}(\mu, V)$ in the competitive equilibrium NODP:

$$w_{NODP}(\mu, V) \triangleq w(\delta, \lambda^{0,\delta}) = \begin{cases} 2V & \text{if } V \geq V(\mu); \\ V^2 \left(1 + \frac{1}{\sqrt{\mu+1}}\right) & \text{if } V < V(\mu). \end{cases} \blacksquare$$

**Proof of Theorem 5.** Suppose in such an equilibrium, the (buy) fill rate in the dark pool is given by $f > 0$, and let $\delta$ be the transaction cost in the open market. For such an equilibrium to exist, it must be the case that a speculator with a high enough signal (in particular, $s = 1$) must be willing to trade in the open market as opposed to the dark pool. This implies that we must have

$$\delta < 1 - f. \quad (24)$$

Similarly, since there are some speculators that enter the dark pool, we must have the following condition between the thresholds of the speculators’ strategies:

$$\frac{1}{1 + f} < \frac{\delta}{1 - f}. \quad (25)$$

Supposing these conditions hold between the transaction cost $\delta$ and the (buy) fill rate $f$, the best response strategies of the traders are as follows: an intrinsic buyer with signal $s > \frac{\delta}{(1+V)(1-f)}$ enters the open market, whereas all other intrinsic buyers enter buy orders in the dark pool. Speculators with signals greater that $\delta/(1-f)$ buy in the open market, those with signals between $1/(1+f)$ and $\delta/(1-f)$ enter buy orders in the dark pool, those with signals between $1 - \delta/(1-f)$ and $f/(1+f)$ enter sell orders in the dark pool, those with signals less than $1 - \delta/(1-f)$ sell in the open market, and finally, the rest do not trade. (The strategy for intrinsic sellers is symmetric.)

Under this strategy profile for the traders, the market maker’s expected utility is given by

$$u(\delta, f) = 2 \left(1 - \frac{\delta}{(1 + V)(1 - f)}\right) \left(\delta - \frac{\delta}{(1 + V)(1 - f)}\right) + 2\mu \left(1 - \frac{\delta}{1 - f}\right) \left(\delta - \frac{\delta}{1 - f}\right).$$

As the market maker sets the transaction cost such that her expected utility is zero, we obtain the following expression for the transaction cost in terms of the buy fill rate:

$$\delta = (1 - f)(1 + V) \left(\frac{(1 + V)(1 + \mu)f - V}{(1 + V)(1 + \mu(1 + V))f - V}\right). \quad (26)$$

Next, note that, in equilibrium, the fill rate $f$ is given by the ratio of the mass of sell orders to that of buy orders in the dark pool, when $\sigma = 1$. Thus, we obtain

$$f = \frac{\mu \left(\left(\frac{1}{1 + f}\right)^2 - \left(1 - \frac{\delta}{1 + f}\right)^2\right) + \left(1 - \left(1 - \frac{\delta}{(1 + V)(1 - f)}\right)^2\right)}{\mu \left(\left(\frac{\delta}{1 + f}\right)^2 - \left(\frac{1}{1 + f}\right)^2\right) + \left(\frac{\delta}{(1 + V)(1 - f)}\right)^2}. \quad (26)$$

Substituting the value of $\delta$ from (26), canceling non-zero factors, and simplifying, we obtain that, in equilibrium, the fill rate must satisfy $g(f, \mu) = 0$, where $g(f, \mu)$ is defined as follows:

$$g(f, \mu) \triangleq f^4 \left((1 + V)^2(1 + \mu)^2(\mu(1 + V)^2 + 1)\right) - f^3 \left(2V(1 + V)(1 + \mu)(\mu(1 + V) + 1)\right)$$

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\[- f^2 \left( \mu^2 (1 + V)^2 (2V^2 + 2V + 1) + 2\mu (1 + V)(V^3 + V^2 + 2V + 1) + 2V + 1 \right) \\
+ f \left( 2V(1 + V)(\mu(1 + V) + 1)) + V^2(\mu V^2 - 1) \right].\]

We observe that \( g(1/\sqrt{\mu}, \mu) < 0 \) and \( g(2/\sqrt{\mu}, \mu) > 0 \) for large enough \( \mu \). Thus, for large enough \( \mu \), there exists a root \( f \in [1/\sqrt{\mu}, 2/\sqrt{\mu}] \) of the polynomial \( g(\cdot, \mu) \).

To conclude there exists an equilibrium, we must verify that the root \( f \) of the polynomial \( g(\cdot, \mu) \) and the corresponding transaction cost in (26) satisfy the necessary conditions in (24) and (25). This is readily verified to be true for all large enough \( \mu \). Thus, for all large enough \( \mu \), there exists a competitive equilibrium, denoted by \( \text{BOTH} \), where both the intrinsic traders and the speculators trade in both the dark pool and the open market.

Let \( f_{\text{BOTH}}(\mu, V) \) denote the buy fill rate, and \( \delta_{\text{BOTH}}(\mu, V) \) denote the transaction cost in the open market in the \( \text{BOTH} \) equilibrium. Since \( f_{\text{BOTH}}(\mu, V) \in [1/\sqrt{\mu}, 2/\sqrt{\mu}] \) for all large enough \( \mu \), we let \( t(\mu, V) = \mu f_{\text{BOTH}}^2(\mu, V) \), with \( t(\mu, V) \in [1, 4] \). Choose any sequence \( \mu_n \to \infty \) such that \( \lim_{n \to \infty} t(\mu_n, V) \) exists, and let the limit be equal to \( t \). As we have \( g(f_{\text{BOTH}}(\mu_n), \mu_n) = 0 \) for all large enough \( \mu_n \), on taking limits as \( n \to \infty \) we obtain that \( t \) satisfies

\[ t^2(1 + V)^4 - t(1 + V)^2(2V^2 + 2V + 1) + V^4 = 0. \]  

(27)

As \( t(\mu_n, V) \in [1, 4] \) for each \( \mu_n \), this implies \( t \) is the unique root of the preceding equation in \([1, 4]\). Since any converging sequence has the same limit, we obtain that \( t(\infty, V) \equiv \lim_{\mu \to \infty} \mu f_{\text{BOTH}}^2(\mu, V) \) exists, and satisfies (27).

This implies that, for all large enough \( \mu \), we have,

\[ f_{\text{BOTH}}(\mu, V) = \sqrt{\frac{t(\infty, V)}{\mu}} + o \left( \frac{1}{\mu} \right). \]

Finally, substituting for \( f_{\text{BOTH}}(\mu, V) \) in the expression for the transaction cost (26), we obtain

\[ \delta_{\text{BOTH}}(\mu, V) = 1 - \left( 1 + \frac{V^2}{(1 + V)^2 t(\infty, V)} \right) \sqrt{\frac{t(\infty, V)}{\mu}} + o \left( \frac{1}{\mu} \right), \]

\[ = 1 - \left( \frac{\sqrt{1 + 2V + 4V^2}}{1 + V} \right) \frac{1}{\sqrt{\mu}} + o \left( \frac{1}{\mu} \right), \]

(28)

where the second equality follows from the fact that \( t(\infty, V) \) satisfies (27).

\[ \textbf{Proof of Theorem 6} \] From (21), we obtain

\[ 1 - \delta_{\text{NODP}}(\mu, V) = \frac{\min \{ V(\mu), V \} }{\sqrt{\mu} + 1}. \]

Similarly, from (28), we have

\[ 1 - \delta_{\text{BOTH}}(\mu, V) = \left( \frac{\sqrt{1 + 2V + 4V^2}}{1 + V} \right) \frac{1}{\sqrt{\mu}} + o \left( \frac{1}{\mu} \right), \]

Since \( V(\mu) = \frac{2\sqrt{\mu + 1}}{\sqrt{\mu + 1}} \to \infty \) as \( \mu \to \infty \), the limit in the theorem statement follows. It is straightforward to verify that the limit is strictly less than one for all \( V \geq 1.428 \), and strictly greater than one for all \( V \in [1, 1.427] \).
Proof of Theorem 7. The welfare in the BOTH equilibrium is given by

\[ w_{BOTH}(\mu, V) = V \left( 1 + \left( 1 - \left( \frac{\delta_{BOTH}(\mu, V)}{(1+V)(1-f_{BOTH}(\mu, V))} \right)^2 \right) \right. 
+ \left. f_{BOTH}(\mu, V) \left( \frac{\delta_{BOTH}(\mu, V)}{(1+V)(1-f_{BOTH}(\mu, V))} \right)^2 \right) 
= V \left( 2 - \frac{1}{(1+V)^2} \right) + o \left( \frac{1}{\sqrt{\mu}} \right). \]

Here the first two terms in the first equality follow from the fact that, when \( \sigma = 1 \), all intrinsic sellers sell the asset, and all intrinsic buyers in the open market buy the asset. The third term in the first equality follows from the fact that only a fraction \( f(\mu, V) \) of the intrinsic buyers in the dark pool end up holding the asset, when \( \sigma = 1 \). All such traders contribute \( V \) to the welfare. (The case when \( \sigma = -1 \) is symmetric.) We obtain the second line after substituting for \( f_{BOTH}(\mu, V) \) and \( \delta_{BOTH}(\mu, V) \) using expressions from the proof of Theorem 5.

Observe that for \( V \geq 2 \), we have \( V(\mu) \leq V \) for all large enough \( \mu \). On the other hand, if \( V < 2 \), then for all large enough \( \mu \), we have \( V < V(\mu) \). Thus, using (22), we obtain the limit in the theorem statement. It is straightforward to verify that the limit is strictly less than one for all \( V \geq 1.88 \), and strictly greater than one for \( V \in [1, 1.879] \).

B. Welfare comparisons for the benchmark case (\( \mu = 0 \))

In this section, we provide the proofs of the welfare comparison results for the benchmark case \( \mu = 0 \) in Section 5.2.

Our results make use of the following lemma, which states that for values of the transaction cost lower than \( \delta_{NODP} \), there is always trade in the open market in any partial equilibrium with a symmetric strategy profile.

Lemma 4. Let \( \mu = 0 \) and \( V > 0 \). For some fixed transaction cost \( \delta \in [0, 1] \), suppose there exists a partial equilibrium in symmetric strategy profile with no trade in the open market. Then, we must have \( \delta \geq \delta_{NODP}(0, V) \).

Proof of Lemma 4. For \( \mu = 0 \), we have \( V(\mu) = 1 \). From (21), for all \( V \geq V(\mu) = 1 \), we have \( \delta_{NODP}(0, V) = 0 \), implying that the lemma statement holds trivially. Hence, for the rest of the proof, assume \( V \in (0, 1) \).

For a fixed \( \delta \in [0, 1] \), consider a partial equilibrium with symmetric strategy profile \( \lambda^{DP} \) where there is no trade in the open market. The strategy \( \lambda^{DP}_{\pm}(\cdot) \) is given by

\[ \lambda^{DP}_{\pm}(s) = \begin{cases} (DP, B) & \text{if } s \geq b_{-}; \\ (DP, S) & \text{if } s \leq a; \\ N & \text{otherwise}, \end{cases} \]

where

\[ b_{-} \triangleq \frac{1 - V}{1 - V + f(1 + V)}, \quad a \triangleq \frac{f(1 - V)}{f(1 - V) + 1 + V}, \]

and \( f \) is the equilibrium fill rate, satisfying the equation

\[ f = \frac{a^2 + (1 - b_{-})^2}{1 - b_{-}^2 + 1 - (1 - a)^2}. \]
Now, observe that
\[ \frac{a^2}{1 - (1 - a)^2} = \frac{a}{2 - a} \leq \frac{a}{1 - a} = \frac{f(1 - V)}{1 + V} \leq f. \]

Thus, since \( f \) satisfies (29), we must have
\[ \frac{(1 - b_-)^2}{1 - b_-^2} \geq f. \]

This implies that
\[ f \leq \frac{1 - b_-}{1 + b_-} = \frac{f(1 + V)}{2(1 - V) + f(1 + V)}. \]

Since \( f > 0 \), we obtain,
\[ (1 + V)(1 - f) \geq 2(1 - V) \geq 1 - V = \delta_{\text{NODP}}(0, V). \]

Finally, observe that since in equilibrium there is no trade in the open market, an intrinsic buyer with signal \( s = 1 \) must prefer submitting a buy order in the dark pool over buying in the open market. Thus, we must have \( u_{\text{OM},B}(V, s) \leq u_{\text{DP},B}(V, s) \) for \( s = 1 \). This implies that, in equilibrium, we have, \( \delta \geq (1 + V)(1 - f) \). Thus, using (30), we obtain \( \delta \geq \delta_{\text{NODP}}(0, V) \).

**Proof of Theorem 3.** For \( \mu = 0 \), we have \( V(\mu) = 1 \). From (21), for all \( V \geq V(\mu) = 1 \), we have \( \delta_{\text{NODP}}(0, V) = 0 \), implying that the theorem statement holds trivially. Hence, for the rest of the proof, assume \( V \in (0, 1) \).

From Lemma 4, we know that for \( \delta < \delta_{\text{NODP}} \), there does not exist a partial equilibrium in symmetric strategy profiles with no trade in the open market. We prove the theorem by considering the two possible types of partial equilibrium for \( \delta < \delta_{\text{NODP}} \), and showing the statement holds for each case.

1. **Case 1. A partial equilibrium with no trade in the dark pool.** Note that, from (23) in the proof of Lemma 3, we obtain that the market maker’s expected utility \( u(\delta, \lambda^{0,\delta}) \) is continuous and strictly increasing in \( \delta \). Since in the competitive equilibrium NODP, we have \( u(\delta_{\text{NODP}}, \lambda^{0,\delta_{\text{NODP}}}) = 0 \), this implies that \( u(\delta, \lambda^{0,\delta}) < u(\delta_{\text{NODP}}, \lambda^{0,\delta_{\text{NODP}}}) = 0 \) for all \( \delta < \delta_{\text{NODP}} \).

2. **Case 2. A partial equilibrium with trade in both the dark pool and the open market.** For \( \delta < \delta_{\text{NODP}}(0, V) = 1 - V \), consider a partial equilibrium \((\delta, \lambda^{f,\delta}) \) with trade in both the dark pool and the open market, where \( f > 0 \) is the (buy) fill rate in the dark pool. Using (19), the strategy \( \lambda^{f,\delta}_V \) has the following structure:

\[
\lambda^{f,\delta}_V(s) = \begin{cases} 
(OM, B) & \text{if } s > x_f(\delta); \\
(DP, B) & \text{if } s \in \left[\frac{1-V}{1-V+f(1+V)}, x_f(\delta)\right]; \\
(DP, S) & \text{if } s \in \left[y_f(\delta), \frac{f(1-V)}{f(1-V)+1+V}\right]; \\
(OM, S) & \text{if } s < y_f(\delta); \\
N & \text{otherwise},
\end{cases}
\]

where \( x_f(\delta) \triangleq \max\{\frac{1-V+f}{2}, \frac{\delta}{(1-V)(1-f)}\} \) and \( y_f(\delta) \triangleq \min\{\frac{1-V-f}{2}, 1 - \frac{\delta}{(1-V)(1-f)}\} \). (The strategy \( \lambda^{f,\delta}_V \) is defined symmetrically.)
Now, the market maker’s expected utility in the partial equilibrium \((\delta, \lambda^f, \delta)\) is given by

\[
u(\delta, \lambda^f, \delta) = 2(\delta - x_f(\delta))(1 - x_f(\delta)) + 2(\delta - 1 + y_f(\delta))y_f(\delta)
= 2(\delta - x_f(\delta))(1 - x_f(\delta)) + 2(\delta - z_f(\delta))(1 - z_f(\delta)),
\]

where \(z_f(\delta) = 1 - y_f(\delta)\). From the definitions, we observe that \(x_f(\delta) \in (\delta, 1]\), and \(y_f(\delta) \in [0, 1 - \delta]\). Furthermore, since under \(\lambda^f, \delta\) there is trade in the open market, either \(x_f(\delta) < 1\) or \(y_f(\delta) > 0\). Taken together, this implies that both \(x_f(\delta)\) and \(z_f(\delta)\) lie in \((\delta, 1]\), with at least one of them strictly less than one. Since the function \((\delta - x)(1 - x)\) is negative for all \(x \in (\delta, 1]\), it follows that \(u(\delta, \lambda^f, \delta) < 0\).

**Proof of Theorem 4.** For \(\mu = 0\), we have \(V(\mu) = 1\). From (21) and (22), for all \(V \geq V(\mu) = 1\), we have \(\delta_{\text{NODP}}(0, V) = 0\) and \(w_{\text{NODP}}(0, V) = 2V\). Since this is the maximum possible welfare under any equilibrium, the theorem statement holds trivially. Hence, for the rest of the proof, assume \(V \in (0, 1]\).

Note that for \(V \in (0, 1)\), we have \(\delta_{\text{NODP}}(0, V) = 1 - V\). In the NODP equilibrium, using (19), the strategy of the intrinsic buyer can be represented as follows:

\[
\lambda^{0,1-V}_{V}(s) = \begin{cases} 
(OM, B) & \text{if } s > b; \\
N & \text{otherwise,}
\end{cases}
\]

where \(b = (1 - V + \delta_{\text{NODP}})/2 = 1 - V\). (The strategy of the intrinsic seller is defined symmetrically.)

For \(\delta \geq \delta_{\text{NODP}}\), in any partial equilibrium \((\delta, \lambda^f, \delta)\) with trade in the dark pool (where \(f\) is the (buy) fill rate in the dark pool), the strategy \(\lambda^f, \delta\) of the intrinsic buyer can be represented using (19) as follows:

\[
\lambda^f, \delta_{V}(s) = \begin{cases} 
(OM, B) & \text{if } s > b_+; \\
(DP, B) & \text{if } s \in (b_-, b_+]; \\
N & \text{if } s \in [a, b_-]; \\
(DP, S) & \text{otherwise,}
\end{cases}
\]

where the thresholds are given by

\[
a = \frac{f(1 - V)}{f(1 - V) + 1 + V}, \quad b_+ = \min \left\{ \max \left\{ \delta, \frac{1 - V + \delta}{2}, 1 \right\}, 1 \right\},
\]

\[
b_- = \min \left\{ \frac{1 - V}{f(1 + V) + 1 - V}, b_+ \right\}.
\]

(The strategy of the intrinsic seller is defined symmetrically.)

These thresholds satisfy one of two conditions:

Case (i) \(b \leq b_- \leq b_+\); OR Case (ii) \(b_- < b \leq b_+\).

In Case (i), it is straightforward to show that the welfare is lower, as fewer intrinsic buyers end up holding the asset, and fewer intrinsic seller end up selling the asset. Hence, hereafter we focus on Case (ii). From \(b_- < b\), we obtain that the fill rate has to satisfy \(f > V/(1 + V)\).

In this case, the change in the welfare from the introduction of the dark pool can be written as

\[
\text{Change in welfare} = F_1(b, b_+)(-V) + F_1(b_-, b_+)f(+V) + F_1(0, a)(-V)
\]
Here the first line corresponds to the net change in welfare when \( \sigma = +1 \), and the second line corresponds to the case when \( \sigma = -1 \). The first term on the first line represents those traders who initially were trading in the open market, but now have decided to enter the dark pool. By forgoing trading in the open market, these traders each contribute a welfare loss of \(-V\). The mass of such traders is \( F_1(b, b+) \). This is offset by the trade in the dark pool: each buyer in the dark pool contributes a welfare gain of \(+V\) with probability \( f \) equal to the fill rate. The mass of such buyers is \( F_1(b_-, b+) \). Finally, we have those intrinsic buyers who were initially not trading, but now have decided to enter an sell order in the dark pool. As \( \sigma = 1 \), these orders are filled with probability one, and the mass of such orders is \( F_1(0, a) \). Each such order contributes a welfare loss of \(-V\). The terms on the second line are obtained in a similar manner.

Rewriting (32), we obtain that the change in welfare is equal to

\[
V \left( -F_1(b, b+) + F_{-1}(b_-, b) - F_1(0, a) \right) + V f \left( F_1(b_-, b+) - F_{-1}(0, a) \right).
\]

Using the fact that \( F_1(x, y) = y^2 - x^2 \) and \( F_{-1}(x, y) = (1-x)^2 - (1-y)^2 \), we obtain

\[
\text{Change in welfare} = V \left( F_1(b) - F_1(b+) + F_{-1}(b) - F_{-1}(b-) - F_1(a) + V f (F_1(b+) - F_1(b) - F_{-1}(a)) \right)
\]

\[
= V \left( b^2 - b^2_+ + 1 - (1-b)^2 - 1 + (1-b)^2 - a^2 \right) + V f \left( b^2_+ - b^2_- - 1 + (1-a)^2 \right)
\]

\[
= V \left( \left( -b^2_+ + 2b - 2b_- + b^2_- - a^2 \right) + f \left( b^2_+ - b^2_- - 2a + a^2 \right) \right).
\]

Next, observe that in a partial equilibrium, the (buy) fill rate \( f \) is given by

\[
f \triangleq \frac{\text{mass of sell orders}}{\text{mass of buy orders}} = \frac{F_1(0, a) + F_{-1}(b-, b_+)}{F_{-1}(0, a) + F_1(b-, b_+)} = \frac{a^2 + 2b_+ - 2b_- - b^2_+ + b^2_-}{2a - a^2 + b^2_+ - b^2_-}.
\]

Thus, letting \( \Delta W \triangleq \text{(change in welfare)}/V \), we obtain

\[
\Delta W = -b^2_+ + 2b - 2b_- + b^2_- - a^2 + \left( \frac{a^2 + 2b_+ - 2b_- - b^2_+ + b^2_-}{2a - a^2 + b^2_+ - b^2_-} \right) (b^2_+ - b^2_- - 2a + a^2).
\]

Observing from (33) that \( b, a, b_-, b_+ \) are functions of \( V, f, \delta \), we define the following functions,

\[
Q(V, f, \delta) \triangleq -b^2_+ + 2b - 2b_- + b^2_- - a^2,
\]

\[
R(V, f, \delta) \triangleq b^2_+ - b^2_- - 2a + a^2,
\]

\[
N(V, f, \delta) \triangleq a^2 + 2b_+ - 2b_- - b^2_+ + b^2_-.
\]

\[
D(V, f, \delta) \triangleq 2a - a^2 + b^2_+ - b^2_-.
\]

(33)

In a partial equilibrium, the fill rate satisfies \( f = N/D \). Moreover, taken as a function of \( V, f, \delta \), we have \( \Delta W = Q + (N/D)R \) for all \( V \in (0, 1) \), \( f \in [V/(1+V), 1) \) and \( \delta \geq \delta_{\text{NODP}} \). Since \( b_- < b \), from Lemma [5] we have \( Q + R \leq 0 \). Now, if \( Q \leq 0 \), we obtain that in any partial equilibrium,

\[
\Delta W = Q + (N/D)R \leq \max\{Q + R, Q\} \leq 0,
\]

where the first inequality follows from the fact that, in any partial equilibrium, the fill rate \( N/D \in [0, 1] \), and the last inequality follows from Lemma [5]. Thus, for the rest of the proof, we further assume that \( Q > 0 \). Then, from Lemma [6] we obtain that \( b_+ < 1 \). This in turn implies that

\[
1 > b_+ = \frac{\delta}{(1-f)(1+V)} = \frac{\delta_{\text{NODP}}}{(1-f)(1+V)} = \frac{1-V}{(1-f)(1+V)},
\]

(34)
from which we obtain that \( f < 2V/(1 + V). \) Since \( f \) also satisfies \( f > V/1 + V, \) we write \( f = V(1 + u)/(1 + V) \) for some \( u \in (0, 1). \) From the definition of the thresholds, \((31)\), we have

\[
a = \frac{f(1-V)}{f(1-V) + 1 + V} = \frac{V(1-V)(1+u)}{1+3V+uV-uV^2},
\]

\[
b_+ = \frac{1-V}{f(1+V)+1-V} = \frac{1-V}{1+uV},
\]

and from \((34)\), we obtain the following lower-bound on \( b_+ \):

\[
b_+ \geq \frac{1-V}{(1-f)(1+V)} = \frac{1-V}{1-uV}.
\]

From Lemma \(7\) we obtain an upper-bound on \( b_+ \), namely, \( b_+ < 2b_+ - 2a^2 + a \). Taken together, we write \( b_+ \) as

\[
b_+ = (1-t)\left(\frac{1-V}{1-uV}\right) + t\left(2b_+ - 2a^2 + a\right)
\]

for some \( t \in [0, 1). \) Using the preceding relations, we can now express \( \Delta W \) as a function of \( V \in (0, 1), \ u \in (0, 1) \) and \( t \in [0, 1). \) However, we can further restrict the domain of \( V \) and \( u. \) To see this, note that we have

\[
\frac{1-V}{1-uV} \leq b_+ < 2b_+ - 2a^2 + a.
\]

Using the expressions for \( b_+ \) and \( a \) from \((35)\), and the fact that \( b = 1-V, \) we obtain

\[
\frac{1-V}{1-uV} < 2(1-V) - \frac{1-V}{1+uV} - 2\frac{V^2(1-V)^2(1+u)^2}{(1+3V+uV-uV^2)^2} + \frac{V(1-V)(1+u)}{1+3V+uV-uV^2}.
\]

Rearranging and canceling non-negative factors, we obtain

\[
0 \geq u^4(2V^5 - 3V^4 + V^3) + u^3(-9V^4 + 8V^3 + 5V^2) + u^2(2V^4 + 19V^3 + 12V^2 + 3V)
\]

\[
+ u(-3V^2 - 1) - 2V^2 - V - 1.
\]

In Lemma \(8\) we show that this implies that \( V \leq \min\left(\frac{37-30u}{25}, 1\right). \) Thus, it suffices to show that

\( \Delta W = Q + (N/D)R \leq 0 \) for \( u \in (0, 1), \ 0 < V \leq \min\left(\frac{37-30u}{25}, 1\right) \) and \( t \in [0, 1). \) Furthermore, since \( D \) denotes the mass of buy orders in the dark pool, we have \( D > 0 \) in any partial equilibrium with trade in the dark pool. Thus, it suffices to show that \( QD + NR \leq 0 \) for \( u \in (0, 1), \ 0 < V \leq \min\left(\frac{37-30u}{25}, 1\right) \) and \( t \in [0, 1). \)

We show this by splitting the analysis into two cases, namely when \(1) u \in (0, 2/5] \) and \( V \in (0, 1]; \)

and \(2) u \in (2/5, 1) \) and \( 0 < V \leq \frac{37-30u}{25}. \)

**Case 1.** \( u \in (0, 2/5] \) and \( V \in (0, 1]: \) We make the following substitution:

\[
u = \frac{2}{5(1+x^2)}, \quad V = \frac{1}{1+y^2}, \quad t = \frac{1}{1+z^2},
\]

where \( x, y, z \in \mathbb{R}. \) On making the substitution, and canceling non-negative factors from the denominator, we are left with a polynomial in \( x^2, y^2, \) and \( z^2 \) with all monomial terms non-positive. From this, we obtain that \( QD + NR \) is non-positive.
Case 2. $u \in (2/5, 1)$ and $0 < V \leq \frac{37 - 30u}{25}$: In this case, we make the following substitution:

$$u = 1 - \frac{3}{5(1 + x^2)}, \quad V = \frac{1}{1 + y^2} \left( \frac{37 - 30u}{25} \right), \quad t = \frac{1}{1 + z^2},$$

where $x, y, z \in \mathbb{R}$. Again, on making the substitution, and canceling non-negative factors from the denominator, we are left with a polynomial in $x^2, y^2,$ and $z^2$ with all monomial terms non-positive. From this, we obtain that $QD + NR$ is non-positive. This completes the proof of the theorem.

The following lemmas are used in the proof of Theorem 4.

**Lemma 5.** For $V \in (0, 1)$ and $\delta \geq \delta_{NODP}(0, V)$, in any partial equilibrium with trade in the dark pool satisfying $b_+ < b$, we have $Q + R \leq 0$.

**Proof.** From (33), we have

$$Q + R = 2b - 2b_+ - 2a.$$ Since, $b_- < b$, we have from (31),

$$b_- = \frac{1 - V}{f(1 + V) + 1 - V}, \quad a = \frac{f(1 - V)}{f(1 - V) + 1 + V}, \quad b = 1 - V.$$

Thus, we get

$$Q + R = 2(1 - V) \left(1 - \frac{1}{f(1 + V) + 1 - V} - \frac{f}{f(1 - V) + 1 + V}\right)$$

$$= -\frac{2V(1 - V)(1 - f)^2(1 + V)}{(f(1 + V) + 1 - V)(f(1 - V) + 1 + V)} < 0.$$  

**Lemma 6.** For $V \in (0, 1)$ and $\delta \geq \delta_{NODP}(0, V)$, in any partial equilibrium with trade in the dark pool satisfying $b_- < b$ and $b_+ = 1$, we have $Q \leq 0$.

**Proof.** Using the definition of $Q$ from (33) for $b_+ = 1$, we have

$$Q \equiv -b_+^2 + 2b - 2b_- + b_+^2 - a^2,$$

$$= -1 + 2b - 2b_- + b_+^2 - a^2,$$

$$= 2b - 2 + (1 - b_-)^2 - a^2,$$

$$= -2V + \frac{f^2(1 + V)^2}{(f(1 + V) + 1 - V)^2} - \frac{f^2(1 - V)^2}{(f(1 - V) + 1 + V)^2}$$

$$= \frac{2V (f^4(1 - V^2)^2 + 4f^3V^2(1 - V^2) + 2f^2(1 + 3V^4) + 4f(1 - V^4) + (1 - V^2)^2)}{(f(1 + V) + 1 - V)^2(f(1 - V) + 1 + V)^2}$$

$$\leq 0,$$

where the fourth equality follows from the definition of the thresholds (31), and the final inequality follows from the fact that the term inside the parenthesis in the numerator is always non-negative.

**Lemma 7.** For $V \in (0, 1)$ and $\delta \geq \delta_{NODP}$, in any partial equilibrium with trade in the dark pool, if $Q > 0$, then we have $b_+ < 2b - b_- - 2a^2 + a$. 

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Proof. Observe that $Q > 0$ implies

$$b^2_+ < 2(b - b_-) + b^2_- - a^2. \tag{37}$$

Furthermore, in a partial equilibrium, we have $N/D \leq 1$. This implies that

$$a^2 + 2b_+ - 2b_- + b^2_+ + b^2_- \leq 2a - a^2 + b^2_+ - b^2_-$$

which leads to

$$b_+ \leq b^2_+ - b^2_- + b_- - a^2 + a$$

$$< (2(b - b_-) + b^- - a^2) - b^2_- + b_- - a^2 + a$$

$$= 2b - b_- - 2a^2 + a,$$

where the second inequality follows from (37).

Lemma 8. For $u, V \in [0, 1]$ with $V > \min((37 - 30u)/25, 1)$, we have

$$u^4(2V^5 - 3V^4 + V^3) + u^3(-9V^4 + 8V^3 + 5V^2) +$$

$$u^2(2V^4 + 19V^3 + 12V^2 + 3V) + u(-3V^2 - 1) - 2V^2 - V - 1 > 0.$$

Proof. The statement is trivially true for $u \leq 2/5$, as in that case $(37 - 30u)/25 \geq 1$. Thus, we only need to consider $u \in (2/5, 1)$ and $V > (37 - 30u)/25$. We make the following change of variables:

$$u = \frac{2 + 3x}{5}, \quad V = (1 - y) + y \left(\frac{37 - 30u}{25}\right) = 1 - \frac{18}{25}xy,$$

where $x \in (0, 1]$ and $y \in [0, 1)$. Substituting, we obtain the polynomial in $x$ and $y$ as

$$P(x, y) = y^5 \left(\frac{-306110016x^6}{6103515625} - \frac{816293376x^8}{6103515625} - \frac{816293376x^7}{6103515625} - \frac{362797056x^6}{6103515625} - \frac{60466176x^5}{6103515625}\right)$$

$$+ y^4 \left(\frac{59521392x^8}{244140625} + \frac{31177872x^7}{244140625} - \frac{49128768x^6}{244140625} - \frac{36531648x^5}{244140625} - \frac{503848x^4}{244140625}\right)$$

$$+ y^3 \left(\frac{-4251528x^7}{9765625} + \frac{1070552x^6}{9765625} - \frac{267688x^5}{9765625} - \frac{2284768x^4}{9765625} - \frac{10054368x^3}{9765625}\right)$$

$$+ y^2 \left(\frac{26244x^6}{78125} - \frac{148716x^5}{78125} + \frac{813564x^4}{78125} + \frac{1241244x^3}{78125} + \frac{335664x^2}{78125}\right)$$

$$+ y \left(-\frac{1458x^5}{15625} + \frac{972x^4}{15625} - \frac{366768x^3}{15625} - \frac{451548x^2}{15625} - \frac{81198x}{15625}\right)$$

$$+ \frac{108x^3}{125} + \frac{1836x^2}{125} + \frac{204x}{125} + \frac{52}{125}.$$
where \( s, t \in \mathbb{R} \). Making the substitution, and writing the polynomial \( P \) as functions of \( s \) and \( t \), we obtain,

\[
(1 + s^2)^5 (1 + t^2)^9 P(t, s) \\
= \frac{4s^{10} (t^2 + 1)^6}{125} \left( 13t^6 + 540t^4 + 1500t^2 + 1000 \right) \\
+ \frac{2s^8 (t^2 + 1)^4}{15625} \left( 16250t^{10} + 666901t^8 + 2853080t^6 + 4570700t^4 + 3169000t^2 + 800000 \right) \\
+ \frac{4s^6 (t^2 + 1)^3}{78125} \left( 81250t^{12} + 3212760t^{10} + 15539976t^8 + 30302525t^6 \\
+ 28784820t^4 + 13201200t^2 + 2317000 \right) \\
+ \frac{4s^4 (t^2 + 1)^2}{9765625} \left( 10156250t^{14} + 386376875t^{12} + 2071716000t^{10} + 4709497408t^8 \\
+ 5564360065t^6 + 3554905650t^4 + 1148445500t^2 + 14283500 \right) \\
+ \frac{4s^2 (t^2 + 1)}{244140625} \left( 126953125t^{16} + 4639484375t^{14} + 2713540000t^{12} \\
+ 69809011025t^{10} + 97143948313t^8 + 77507176490t^6 \\
+ 34774505800t^4 + 7864505000t^2 + 65425000 \right) \\
+ \frac{1}{6103515625} \left( 2539062500t^{18} + 88985156250t^{16} + 559961250000t^{14} \\
+ 1594118520000t^{12} + 25182821688000t^{10} + 2354402950374t^8 \\
+ 1295555779740t^6 + 387607598400t^4 + 49432614000t^2 + 35054000 \right).
\]

From this, it follows that \( P(t, s) > 0 \) for all \( s, t \in \mathbb{R} \). Hence \( P(x, y) > 0 \) for all \( x \in [0, 1] \) and \( y \in [0, 1) \). This completes the proof. ■

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