Information Aggregation and Allocative Efficiency in Smooth Markets

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Recent years have seen extensive investigation of the information aggregation properties of markets. However, relatively little is known about conditions under which a market will aggregate the private information of rational risk averse traders who optimize their portfolios over time; in particular, what features of a market encourage traders to ultimately reveal their private information through trades? We consider a market model involving finitely many informed risk-averse traders interacting with a market maker. Our main result identifies a basic *asymptotic smoothness* condition on prices in the market that ensures information is aggregated as long as portfolios remain bounded; further, under this assumption, the allocation achieved is *ex post* Pareto efficient. Asymptotic smoothness is fairly mild: it requires that, eventually, infinitesimal purchases or sales should see the same per unit price. Notably, we demonstrate that, under some mild conditions, algorithmic markets based on cost function (or, equivalently, markets based on market scoring rules) aggregate the information of traders.

Key words: Information Aggregation ; Allocative Efficiency ; Asymptotic Smoothness ; Risk Aversion

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1. Introduction Recent years have seen a surge of development and interest in *prediction markets*. These are typically online markets where the assets pay a fixed amount if a given event occurs. The basic goal of a prediction market is to determine the likelihood that a given event will happen: informally, if the price of an event is low relative to the likelihood that it will occur, then at least some traders should find it profitable to buy—and thus the price will go up. In this way, the expectation is that prices in prediction markets *aggregate* the information available to individual traders [15].

This basic goal has led to extensive investigation of the information aggregation properties of prediction markets. More broadly, however, the notion that information is aggregated in market prices has been a longstanding topic of study in economics; for example, see [7, 10, 11, 16]. Roughly speaking, the idea that market prices should reflect the information collectively possessed by market participants is one element of the *efficient markets hypothesis* [9]. A second, closely related objective is that markets that function well should achieve *ex post* Pareto efficient allocations: it should not be possible to reallocate assets between agents so that some agent's posterior expected utility is improved without reducing the posterior expected utility of another agent. Taken together, these goals—*information aggregation* and *allocative efficiency*—underlie much of the appeal of decentralized markets in settings where traders possess asymmetric information.

Despite the central connection between information aggregation, *ex post* Pareto efficiency, and market behavior, significant open problems remain. In particular, relatively little is known about conditions under which a market will aggregate the private information of rational risk-averse traders who optimize their portfolios over time, or whether the market will achieve efficient allocations. The central goal of this paper is to investigate conditions under which these results are obtained.

Our main contributions are as follows:

(i) *Smoothness and information aggregation.* Our main result identifies a basic *smoothness* condition on prices in a complete market that ensures information will be aggregated. Formally, we show that if the portfolios held by traders remain bounded over time, and the limiting price charged by the market maker is *continuously differentiable* at zero with respect to the quantity traded, then the market will aggregate information in any perfect Bayesian equilibrium. This is an interesting result, because, in particular, it suggests the per unit price for a small purchase and a small sale should be essentially the same—that is, there should be no bid-ask spread near zero quantity.

We note that without such a smoothness condition, it is possible that information aggregation may fail. In particular, if a strategic market maker is sufficiently concerned that traders possess significantly superior information regarding the value of a traded asset, then prices may be set in a manner that completely precludes trading and thus the dissemination of information [7]. Our main contribution is to demonstrate that a smoothness assumption on the market ensures sufficient trade and thereby guarantees the diffusion of the information available to the individual traders. Notably, many algorithmic market makers used in prediction markets *are* smooth in the sense we require; in particular, we give conditions under which *cost function* market makers (or, equivalently, market makers based on market scoring rules) satisfy the smoothness requirement.

(ii) *Markets with risk averse traders.* The market model we consider involves finitely many informed *risk-averse* traders interacting with a market maker. This is distinct from prior work in this area [4, 5, 14], which primarily considered information aggregation among traders that are risk neutral.

This modeling choice is significant for two reasons. First, in many markets, risk aversion is important. For example, traders may bid on contracts in prediction markets as insurance to hedge against risks inherent in other elements of their portfolio. Second, if all traders and the market maker are risk neutral and strategic, then in general, the *no-trade theorem* applies and precludes trade [13]. Informally, the problem is that two rational risk neutral traders cannot take opposing positions in the market and both expect to be better off. However, with risk aversion, trading can occur even if all traders are rational: traders may trade purely on the motive of hedging.

(iii) *Allocative efficiency.* We show that regardless of the level of risk aversion of the traders, the final allocation and prices together constitute a *competitive equilibrium*; thus, in particular, the final portfolios of the traders are *ex post* Pareto efficient. Note that as a consequence of this result, when traders are risk averse, prices may not reflect the true posterior probabilities of events occurring. This is because competitive equilibrium prices must also reflect the marginal expected utility of the traders; thus, in general, prices are *risk-adjusted* probabilities. If even one trader in the market is risk neutral, however, then prices are accurate posterior probabilities.

The most closely related paper to our work is by Ostrovsky [14]. He shows that if the securities under consideration are "separable" in an appropriate sense, and all traders are risk neutral, then information is aggregated in markets based on the competitive dealer model of Kyle [11], as well as in markets based on market scoring rules. The markets we consider are complete with respect to payoff-relevant uncertainty (i.e., there exists one contingent contract for each possible payoff-relevant event that can occur); and with the partition model for signal structure that Ostrovsky considers, a complete market is always separable.[1] Our main innovation is in studying markets with risk averse traders, and establishing the aforementioned smoothness condition as essential to information aggregation.

The remainder of the paper is organized as follows. In the next section we define our basic model, including the game played by market participants, as well as perfect Bayesian equilibrium for this game. In Section 3, we formally define information aggregation. In Section 4, we claim that if the portfolios of traders remain bounded and the market is asymptotically smooth, then information is aggregated. In Section 5, we provide insight into *ex post* Pareto efficiency of the market. In Section 6, we study a class of markets based on cost functions, and show information aggregation under the assumption of portfolio boundedness, if the loss to the market maker is bounded; the latter condition is satisfied by a wide range of cost functions. Finally, in Section 7 we extend our results to a more general signal structure.

2. Model In this section we describe the operation of the market, as well as our equilibrium notion for the resulting game, perfect Bayesian equilibrium.

2.1 Market Operation We consider a market consisting of n traders and organized by a market maker. Trading takes place in the market sequentially at an infinite sequence of times $t \in \{1, 2, \ldots\}$. In particular, at time t the trader i_t engages in trade with the market maker, and the sequence $\{i_t\}$ is known *a priori* to all the traders and the market maker. We further assume that each trader visits the market infinitely often.

The uncertainty in the value of future securities is captured by a random variable $\tilde{\omega}$ taking values in the finite set $\Omega = \{1, \ldots, m\}$; we refer to the random variable $\tilde{\omega}$ as *the payoff-relevant state* of the world, and assume that all traders have a common prior distribution for $\tilde{\omega}$. We assume that the payoff-relevant state is only revealed after all trades are completed. Further, we assume the market for securities over payoff-relevant uncertainty, i.e., possible realizations of $\tilde{\omega}$, is *complete*. That is, we assume traders can trade in any of m securities labeled $1, \ldots, m$; one share of security ω pays \$1 in state ω, and 0 otherwise. Note that we do not make the assumption of complete markets over *all* sources of uncertainty; in particular, traders do not trade over *payoff-irrelevant* events that do not directly affect the trader's preferences.

Suppose that trader $i_t = i$ visits the market at time t to trade with the market maker. Let $y_t \in \mathbb{R}^m$ denote the corresponding trade, where the component $y_t(\omega)$ is the quantity of security ω

[1] Although in the case of finite signal space, our model of signals can be reformulated as a partition model, our results hold in more general signal spaces. Moreover, Ostrovsky assumes that upon pooling of all the signals of the traders, there is no further uncertainty in the market, whereas we allow for payoff-relevant uncertainty even after the signals of the traders are pooled.

bought by the trader i at time t. The history h_t at time t as observed by the traders (and the market maker) consists of all the trades until time t. In other words:

$$h_t \triangleq (y_1, y_2, \ldots, y_{t-1}).$$

As a matter of convention, we let $h_1 \triangleq \emptyset$ denote the null history at time 1. We denote by H_t the set of all possible histories up to time t, and let $H_f \triangleq \cup_{t\geq 1} H_t$ denote the set of all possible finite histories. Finally, let $h_\infty \triangleq (y_1, y_2, \ldots)$ denote the infinite history, or the path taken by the market, and let H_∞ denote the set of all possible infinite histories.

The portfolio of a trader i at time t consists of the different quantities of each security she holds. Let $w_{i,t}(\omega)$ denote the quantity of security ω held by trader i at time t; we refer to the vector $w_{i,t}$ as the *portfolio* of trader i at time t. We assume that the initial portfolio of each trader is common knowledge among the traders and the market maker.

Observe that if a trader holds the portfolio $\mathbf{1} \triangleq (1, \ldots, 1)$, i.e., one unit of each security, then the trader receives a payoff of \$1 regardless of the realized state. For this reason we refer to the $\mathbf{1}$ portfolio as *money*, and throughout the paper we interpret monetary payment of \$$x$ to or from a trader as credits or debits of x units of the $\mathbf{1}$ portfolio.

The market maker determines the price for trades of different quantities of the securities; this price may depend on the history. In particular, we let $K(h_t, y)$ represent the price charged for a portfolio y after history h_t; thus the trader's net trade at time t is $y_t - K(h_t, y_t)\mathbf{1}$. We assume the functional form of K is known to all traders *a priori*. (We note that this does not preclude a strategic market maker: our model allows for the possibility that K is simply the equilibrium pricing strategy of the market maker.) To reflect the fact that the trade in the market is voluntary, we assume that the pricing function K does not penalize any trader for not participating in the market. More precisely, we assume that after any history h_t, the pricing function satisfies $K(h_t, \mathbf{0}) = 0$, where $\mathbf{0} \triangleq (0, \ldots, 0)$.

2.2 The Game and Equilibrium In this section our main goal is to define our equilibrium concept, *perfect Bayesian equilibrium (PBE)*. Informally, PBE requires that traders' strategies maximize their utility given their beliefs over any uncertain elements of the model, and their beliefs are consistent with the strategies adopted by other traders in equilibrium. In our model, uncertainty arises because the payoff-relevant state $\tilde{\omega}$ is unknown; however, we assume traders are informed, and receive signals regarding the true state. In this section we define signals, beliefs, strategies, utilities, and, ultimately, the concept of PBE.

It should be noted that, to be precise, these constructs should be defined in a measure-theoretic framework. However, for clarity of exposition we suppress measure-theoretic details in the main text; in Appendix B we provide a formal measure-theoretic description of each of the elements introduced here.

Signals. Before trading begins, each trader i receives a private signal $\tilde{s}_i \in \Sigma_i$. We assume that the private signals are independent conditional on the payoff-relevant state $\tilde{\omega}$; we revisit and relax this assumption in Section 7. Further, we assume that the joint distribution of $\tilde{\omega}$ and the signals is common knowledge among the traders. Let $\mathbf{P}$ denote the joint prior distribution of $\tilde{\omega}$ and $\tilde{s}_1, \ldots, \tilde{s}_n$. We refer to $\tilde{s} \triangleq (\tilde{s}_1, \ldots, \tilde{s}_n)$ as the joint signal and $\Sigma \triangleq \Sigma_1 \times \cdots \times \Sigma_n$ as the joint signal space.[2]

[2]A special case of our general model is the *finite partition model* of signals. This case arises when each Σ_i is a

Beliefs. Let $\mathcal{S} \triangleq H_\infty \times \Omega \times \Sigma$. Observe that an element of $\mathcal{S}$ captures all uncertainty in the model: the path of the market, the payoff-relevant state, and the signals of all the traders. All uncertainty can therefore be represented by probability distributions over this space. In particular, we assume that after each history h_t, and having observed signal s_i, trader i's *belief* $\nu_i(h_t, s_i)$ is a probability distribution over $\mathcal{S}$. This represents trader i's forecast of the future actions by traders, the payoff-relevant state $\tilde{\omega}$, and the signal vector $\tilde{s}$.

Strategies. The trader i is said to follow the strategy δ_i if, after a history $h \in H_f$, the trader selects a trading decision according to the distribution specified by $\delta_i(h, s_i)$, where s_i denotes the private signal received by the trader. (Of course, a trader i can only trade at those times where $i_t = i$.) Let $y_t(\delta_i, h_t)$ denote the trade specified by the strategy δ_i at time t. If δ_i is a mixed strategy, then $y_t(\delta_i, h_t)$ is the realized trade when trader i chooses her trade according to δ_i. Note that the distribution of $y_t(\delta_i, h_t)$ at time t depends only on the realized history h_t and the private signal $\tilde{s}_i$. When the context is clear, we omit the explicit history dependence. Let $\delta = (\delta_1, \ldots, \delta_n)$ denote the collective strategy profile, where each trader i follows the strategy δ_i. Also, for each i, let $\delta_{-i} = (\delta_j)_{(j \neq i)}$ denote the strategy of every trader except i.

Note that the strategy profile δ, together with the common prior distribution $\mathbf{P}$ over the payoff-relevant state of the world and the signals, defines a probability measure Q^δ over the space $\mathcal{S}$.

Utility and the value function. We assume that if trader i is holding a portfolio w_i, and the payoff-relevant state $\tilde{\omega} = \omega$ is realized, then trader i obtains a utility $u_i(\omega, w_i(\omega))$. The utility function $u_i \colon \Omega \times \mathbb{R} \to \mathbb{R}$ of each trader is common knowledge among the traders and the market maker. We consider traders who are risk averse, and who, all else being equal, prefer greater wealth to less wealth. These two characteristics of the traders are captured by assuming that, for each trader i and each payoff-relevant state ω, the function $u_i(\omega, \cdot)$ is strictly increasing and concave. We further make the assumption in our analysis that each function $u_i(\omega, \cdot)$ is differentiable. Note that all these assumptions hold trivially for risk neutral traders, whose utility functions are given by $u_i(\omega, x) = x$ for all $\omega \in \Omega$, $x \in \mathbb{R}$.

While we assume that traders act as expected utility maximizers, as trading proceeds over an infinite horizon, defining this objective precisely is somewhat subtle. In particular, if the history up to time t is h_t, the strategy profile is δ, and trader i's belief is ν_i, then trader i's *value function* at time t is defined to be

$$J_{i,t}(\delta) \triangleq \mathbf{E}_{\nu_i(h_t, s_i), \delta}\left[\liminf_{T \to \infty} u_i(\tilde{\omega}, w_{i,T}(\tilde{\omega}))\right]. \tag{1}$$

We assume that traders act at each time period to maximize their value function. To motivate this assumption on trader behavior, suppose for a moment that the portfolio of the trader converges over time. Then, the quantity $\lim_{T \to \infty} u_i(\tilde{\omega}, w_{i,T}(\tilde{\omega}))$ is the final utility of the trader i at the conclusion of trading in the market. Thus the value function captures the expected utility of the eventual portfolio of the trader. Defining the value function in terms of the limit inferior is a generalization that allows us to capture a broader class of possible trader behaviors that is not restricted to only those cases where the portfolios are required to converge asymptotically. This approach is common, for example, in the literature on infinite horizon dynamic programming problems. Note

partition of the payoff-relevant state space Ω, and the private signal $\tilde{s}_i$ denotes the partition element in which $\tilde{\omega}$ is contained. Conversely, it can be shown that any general signal structure where the signal space Σ_i of trader i is finite can be reformulated as a finite partition model. However, such a reformulation involves the introduction of additional components in the state, altering the state space Ω and possibly rendering a complete market incomplete.

that the value function is not time separable over the trades at different time periods, and there is no discounting involved for trades occurring at later periods. We assume that for all cases of interest, the value functions are well-defined at all times for all traders. In particular, we assume that the expectation in (1) exists (but is not necessarily finite) for all times and for all traders.

Perfect Bayesian equilibrium. Finally, we describe the notion of PBE for the market defined above. We say the strategy profile δ and the beliefs ν_i for each i constitute a PBE if the following two conditions hold :

(i) For each i, for each $h_t \in H_t$ and all $s_i \in \Sigma_i$, we have

$$\delta_i \in \operatorname*{argmax}_{\delta_i'} J_{i,t}(\delta_i', \delta_{-i}).$$

(ii) For each i and for each s_i, the belief $\nu_i(h_t, s_i)$ after any history $h_t \in H_t$ is derived from the belief $\nu_i(h_{t-1}, s_i)$ after history h_{t-1} through Bayes' rule whenever possible.

The first condition requires that after any history, the action specified by the strategy δ_i for trader i is optimal, holding fixed the strategies of all the other traders. The second condition requires that after any history, the traders update their beliefs through Bayes' rule whenever possible. Note that, along an equilibrium path, we have $\nu_i(h_t, s_i)(\cdot) = Q^\delta(\cdot \mid h_t, \tilde{s}_i = s_i)$ for all $s_i \in \Sigma_i$.

Whenever we consider a PBE associated with a particular strategy profile δ, we append δ as a superscript to all the relevant quantities. Thus, for example, $w_{i,t}^\delta$ denotes the portfolio of trader i at time t in the PBE with strategy profile δ.

3. Information Aggregation

In this section, we define the notion of information aggregation. Informally, information aggregation occurs when the private signals observed by individual traders are aggregated into a common market belief. In order to give a precise definition, we begin by defining some notation.

Fix a PBE with strategy profile δ and beliefs $\{\nu_i\}$. Recall that Q^δ is the joint probability measure induced on the space $\mathcal{S}$ by the strategy profile δ and the common prior distribution $\mathbf{P}$ over the payoff-relevant state and the signals. For much of our analysis, we require notation for the belief of an uninformed market observer who is able to observe market transactions, is aware of all common knowledge, and shares the same prior distribution as market participants. In particular, we define φ_t, the *common belief of the market* over the payoff-relevant state and private signals at time t, by

$$\varphi_t(\cdot) \triangleq Q^\delta(\cdot|h_t).$$

The notion of a common belief is an important tool in analyzing dynamic games with asymmetric information in a number of different contexts, for example, in models of rational learning [2], herding [17], and reputation [12]. A similar idea was employed by Ostrovsky in his work on information aggregation [14].

Further, trader i has beliefs that are informed by both the history of trading and the private signal observed. We define $\varphi_{i,t}$ to be the belief of trader i over the payoff-relevant state and signals at time t, and in a PBE this is given by

$$\varphi_{i,t}(\cdot) \triangleq Q^\delta(\cdot|h_t, \tilde{s}_i = s_i).$$

Note that φ_t and $\varphi_{i,t}$ have implicit history dependence. Moreover, trivially, $\varphi_{i,t}$ can be obtained from φ_t by conditioning on trader i's signal: $\varphi_{i,t}(\cdot) = \varphi_t(\cdot|\tilde{s}_i = s)$.

The following result is common in analysis of PBE in infinite horizon games: it establishes that beliefs converge to well-defined limits. The proof, which is omitted, involves writing the probabilities as a Doob martingale and then applying the martingale convergence theorem.

LEMMA 3.1 *Under the measure Q^δ, almost surely, the sequence of beliefs φ_t (resp., $\varphi_{i,t}$) converges weakly to a probability distribution φ_∞ (resp., $\varphi_{i,\infty}$), where $\varphi_\infty(\cdot) = Q^\delta(\cdot|h_\infty)$, and $\varphi_{i,\infty}(\cdot) = Q^\delta(\cdot|h_\infty, \tilde{s}_i = s_i)$.*

We are now ready to make the main definition.

DEFINITION 3.1 ***(Information Aggregation)*** *The market aggregates the information of the traders if, almost surely, for all $\omega \in \Omega$,*

$$\varphi_\infty(\tilde{\omega} = \omega) = \mathbf{P}(\tilde{\omega} = \omega|\tilde{s}).$$

On the left hand side, we have the posterior common belief of the market after the infinite trading history has been observed. On the right hand side, we have the posterior distribution of the payoff-relevant state if all traders' signals could be pooled. Thus, information aggregation requires that, via the trading history, the common market belief completely pools the private signals of the traders.

Note that the preceding definition does not require that the prices of the securities reflect the posterior beliefs of the traders. Our definition only requires that an uninformed outsider sharing the common knowledge and prior distribution of the traders, and having knowledge of the sequence of trades conducted, should be able to infer the relevant information in the joint private signal $\tilde{s}$. Subsequently, we show a stronger result: in fact, the asymptotic portfolios and prices together constitute a *competitive equilibrium* of the limiting economy, where traders maximize expected utility given the joint private signal $\tilde{s}$. This observation allows us to show that an uninformed outsider can infer the information in the joint private signal under much milder requirements. Further, we demonstrate the relationship between prices and the utility functions of the traders.

4. Asymptotic Smoothness

A central theme of our paper is that if the prices set by the market maker are sufficiently "smooth" with respect to small purchases or sales by traders, then information will be aggregated. Formally, we introduce the following condition:

ASSUMPTION 4.1 ***(Asymptotic Smoothness)*** *Consider a strategy profile δ and the associated induced distribution Q^δ. We assume that there exists an open neighborhood $\mathcal{N}$ of zero such that almost surely under the distribution Q^δ, the limit*

$$K(h_\infty, y) \triangleq \limsup_{t\to\infty} K(h_t, y),$$

is finite and continuously differentiable at all $y \in \mathcal{N}$.

The preceding condition essentially requires that, asymptotically, if a trader buys or sells an infinitesimal portfolio, the marginal price is the same. This rules out the possibility of a nonzero bid-ask spread for infinitesimal trades. Note that such nonzero bid-ask spreads have been observed in prior models, e.g., in the limit-order book model with adverse selection as studied by Glosten [6]. A central observation in those models is that if the initial bid-ask spread is too wide, then trading may not take place simply because informed traders will not find it profitable to participate. Using this insight it is straightforward to construct examples of markets that are not asymptotically smooth, and in which information aggregation does not occur.

Although we have stated asymptotic smoothness as a property of the pricing function K, it may alternatively be viewed as the property of the underlying *equilibrium.* In other words, one might consider equilibria where the behavior of the market marker arises endogenously. In such settings, if the induced pricing function K that arises from the market maker's activity is asymptotically smooth, then our results apply.

Additionally, we make the following assumption on the equilibria under consideration.

Assumption 4.2 ***(Portfolio Boundedness)*** *Under the strategy profile δ, the portfolios of each trader i satisfy*

$$-\infty < \underline{w}_{i,\infty}^{\delta}(\omega) \triangleq \liminf_{t\to\infty} w_{i,t}^{\delta}(\omega) < +\infty,$$

almost surely for all $\omega \in \Omega$.

Portfolio boundedness precludes those instances where the market maker's pricing function is too weak to restrict traders from taking arbitrarily large positions. The assumption is violated, for example, in instances where a trader possesses a sequence of actions which yield unbounded utility. Furthermore, the assumption also captures the idea that the traders are unwilling to accept unbounded losses in *any* state, even states on which their belief assigns zero weight. In these respects, one can view Assumption 4.2 as excluding pathological equilibria.

The following theorem, which is the main result of this section, shows information is aggregated if asymptotic smoothness and portfolio boundedness hold.

Theorem 4.1 *In any PBE where asymptotic smoothness (Assumption 4.1) and portfolio boundedness (Assumption 4.2) hold, the market aggregates the information of the traders.*

Before we prove the theorem, we provide some intuition for the result. In a PBE, each trader starts with private information which may not be reflected in the pricing function of the market maker. As long as the information of any trader is not fully incorporated into the pricing function, asymptotic smoothness (Assumption 4.1) suggests that, eventually, the trader should be able to profit from trading infinitesimal quantities of the appropriate securities. Portfolio boundedness in the PBE (Assumption 4.2), on the other hand, suggests that, eventually, the trader ceases to exploit any such opportunities. We therefore expect that all the information of an individual trader will ultimately be incorporated into the prices, and, hence, that information aggregation will occur.

In order to prove Theorem 4.1, we need the following lemma; the proof can be found in Appendix A. It makes the intuitive observation that under Q^{δ}, even after observing past history, the signals are independent conditional on the payoff-relevant state. (Note that this result holds even after infinite history, i.e., $t = \infty$).

Lemma 4.1 *Under Q^{δ}, conditional on the history h_t and payoff-relevant state $\tilde{\omega}$, the private signals $\tilde{s}_1, \ldots, \tilde{s}_n$ are independent, for any $1 \leq t \leq \infty$.*

The proof of Theorem 4.1 uses the fact that in a PBE, a unilateral deviation by any trader should result in lower utility for that trader. Using this fact, we show that in the limit, the trader i's belief over $\tilde{\omega}$, given the infinite history h_∞, is independent of her private signal $\tilde{s}_i$. We then obtain the result via Lemma A.1 in Appendix A, a basic lemma on independent random variables.

Proof of Theorem 4.1. Given the continuity of the utility function, the value function of a trader i after the history h_t can be written as

$$J_{i,t}(\delta) = \mathbf{E}_{\nu_i(h_t,\tilde{s}_i)}\left[u_i(\tilde{\omega}, \underline{w}^{\delta}_{i,\infty}(\tilde{\omega}))\right].$$

Now, consider a unilateral deviation for trader i after any history h_t, where the trader instead of following the strategy δ_i, decides to trade a fixed quantity $z \in \mathbb{R}^m$ and never trade in the market thereafter. Let $w^z_{i,t}$ denote the payoff vector of the trader i after such a trade, i.e.,

$$w^z_{i,t} \triangleq w^{\delta}_{i,t-1} + z - K(h_t, z)\mathbf{1}.$$

As δ is a PBE strategy profile, any deviation after history h_t cannot lead to a higher utility for the trader. Thus, almost surely,

$$\mathbf{E}_{\nu_i(h_t,\tilde{s}_i)}\left[u_i(\tilde{\omega}, w^z_{i,t}(\tilde{\omega}))\right] \le \mathbf{E}_{\nu_i(h_t,\tilde{s}_i)}\left[u_i(\tilde{\omega}, \underline{w}^{\delta}_{i,\infty}(\tilde{\omega}))\right], \tag{2}$$

for all $z \in \mathbb{R}^m$, and for all t. Next, observe that

$$\mathbf{E}_{\nu_i(h_t,\tilde{s}_i)}\left[u_i(\tilde{\omega}, w^z_{i,t}(\tilde{\omega}))\right] = \sum_{\omega\in\Omega} \varphi_{i,t}(\tilde{\omega} = \omega) u_i(\omega, w^{\delta}_{i,t}(\omega) + z(\omega) - K(h_t, z)).$$

Taking limits, we obtain, almost surely,

$$\begin{aligned}\liminf_{t\to\infty} \mathbf{E}_{\nu_i(h_t,\tilde{s}_i)}\left[u_i(\tilde{\omega}, w^z_{i,t}(\tilde{\omega}))\right] &= \liminf_{t\to\infty} \sum_{\omega\in\Omega} \varphi_{i,t}(\tilde{\omega} = \omega) u_i(\omega, w^{\delta}_{i,t}(\omega) + z(\omega) - K(h_t, z)) \\ &\ge \sum_{\omega\in\Omega} \liminf_{t\to\infty} \left[\varphi_{i,t}(\tilde{\omega} = \omega) u_i(\omega, w^{\delta}_{i,t}(\omega) + z(\omega) - K(h_t, z))\right].\end{aligned}$$

Here, the inequality follows from the fact that the limit inferior of a sum of finitely many terms is greater than the sum of the limit inferiors of the summands. Similarly, observe that, by asymptotic smoothness (Assumption 4.1) and portfolio boundedness (Assumption 4.2),

$$\liminf_{t\to\infty}\left[w^{\delta}_{i,t}(\omega) + z(\omega) - K(h_t, z)\right] \ge \underline{w}^{\delta}_{i,\infty}(\omega) + z(\omega) - K(h_\infty, z).$$

It follows that the limit on the left hand side is bounded below almost surely for all $\omega \in \Omega$ and $z \in \mathcal{N}$. Furthermore, we know from Lemma 3.1 that for each $\omega \in \Omega$, the trader's belief $\varphi_{i,t}(\tilde{\omega} = \omega)$ converges to $\varphi_{i,\infty}(\tilde{\omega} = \omega)$ almost surely. Using these facts and the strict monotonicity and continuity of the utility function, we obtain, almost surely,

$$\begin{aligned}\liminf_{t\to\infty} \mathbf{E}_{\nu_i(h_t,\tilde{s}_i)}\left[u_i(\tilde{\omega}, w^z_{i,t}(\tilde{\omega}))\right] &\ge \sum_{\omega\in\Omega} \liminf_{t\to\infty}\left[\varphi_{i,t}(\tilde{\omega} = \omega) u_i(\omega, w^{\delta}_{i,t}(\omega) + z(\omega) - K(h_t, z))\right] \\ &= \sum_{\omega\in\Omega} \varphi_{i,\infty}(\tilde{\omega} = \omega) u_i\left(\omega, \liminf_{t\to\infty}\left[w^{\delta}_{i,t}(\omega) + z(\omega) - K(h_t, z)\right]\right) \\ &\ge \sum_{\omega\in\Omega} \varphi_{i,\infty}(\tilde{\omega} = \omega) u_i\left(\omega, \underline{w}^{\delta}_{i,\infty}(\omega) + z(\omega) - K(h_\infty, z)\right) \\ &= \mathbf{E}_{\varphi_{i,\infty}}\left[u_i(\tilde{\omega}, w^z_{i,\infty}(\tilde{\omega}))\right],\end{aligned}$$

where $w^z_{i,\infty}$ is defined as

$$w^z_{i,\infty} \triangleq \underline{w}^\delta_{i,\infty} + z - K(h_\infty, z)\mathbf{1}.$$

Thus, almost surely,

$$\mathbf{E}_{\varphi_{i,\infty}}\left[u_i(\tilde{\omega}, w^z_{i,\infty}(\tilde{\omega}))\right] \leq \liminf_{t\to\infty} \mathbf{E}_{\nu_i(h_t,\tilde{s}_i)}\left[u_i(\tilde{\omega}, w^z_{i,t}(\tilde{\omega}))\right]. \tag{3}$$

On the other hand, almost surely,

$$\begin{aligned}\lim_{t\to\infty} \mathbf{E}_{\nu_i(h_t,\tilde{s}_i)}\left[u_i(\tilde{\omega}, \underline{w}^\delta_{i,\infty}(\tilde{\omega}))\right] &= \lim_{t\to\infty} \mathbf{E}_{Q^\delta}\left[u_i(\tilde{\omega}, \underline{w}^\delta_{i,\infty}(\tilde{\omega}))\ |h_t, \tilde{s}_i\right] \\ &= \mathbf{E}_{Q^\delta}\left[u_i(\tilde{\omega}, \underline{w}^\delta_{i,\infty}(\tilde{\omega}))\ |\ h_\infty, \tilde{s}_i\right] \\ &= \mathbf{E}_{\varphi_{i,\infty}}\left[u_i(\tilde{\omega}, \underline{w}^\delta_{i,\infty}(\tilde{\omega}))\right]. \end{aligned}\tag{4}$$

Here, the first equality follows since ν_i is the belief of trader i in a PBE. The second equality follows by a backwards martingale convergence argument. The third equality follows by the definition of $\varphi_{i,\infty}$.

Thus, from equations (2), (3) and (4), we obtain almost surely for all $z \in \mathcal{N}$ that

$$\begin{aligned}\mathbf{E}_{\varphi_{i,\infty}}\left[u_i(\tilde{\omega}, w^z_{i,\infty}(\tilde{\omega}))\right] &\leq \liminf_{t\to\infty} \mathbf{E}_{\nu_i(h_t,\tilde{s}_i)}\left[u_i(\tilde{\omega}, w^z_{i,t}(\tilde{\omega}))\right] \\ &\leq \liminf_{t\to\infty} \mathbf{E}_{\nu_i(h_t,\tilde{s}_i)}\left[u_i(\tilde{\omega}, \underline{w}^\delta_{i,\infty}(\tilde{\omega}))\right] \\ &= \mathbf{E}_{\varphi_{i,\infty}}\left[u_i(\tilde{\omega}, \underline{w}^\delta_{i,\infty}(\tilde{\omega}))\right]. \end{aligned}$$

Since $w^{\mathbf{0}}_{i,\infty} = \underline{w}^\delta_{i,\infty}$, we conclude that, almost surely,

$$\mathbf{0} \in \operatorname*{argmax}_{z\in\mathcal{N}}\ \mathbf{E}_{\varphi_{i,\infty}}\left[u_i(\tilde{\omega}, w^z_{i,\infty}(\tilde{\omega}))\right].$$

By asymptotic smoothness, the pricing function is $K(h_\infty, y)$ is differentiable at $y = \mathbf{0}$. Moreover, the traders have differentiable utility functions. This implies the following first order necessary condition holds almost surely, for each $\omega \in \Omega$:

$$\varphi_{i,\infty}(\tilde{\omega} = \omega) u'_i(\omega, \underline{w}^\delta_{i,\infty}(\omega)) = \nabla_{y(\omega)} K(h_\infty, \mathbf{0})\ \mathbf{E}_{\varphi_{i,\infty}}\left[u'_i(\tilde{\omega}, \underline{w}^\delta_{i,\infty}(\tilde{\omega}))\right].$$

Recall that $\varphi_{i,\infty}(\tilde{\omega} = \omega) = \varphi_\infty(\tilde{\omega} = \omega \big| \tilde{s}_i)$. Using this fact and rearranging the above equation, for each $\omega \in \Omega$ we have almost surely that

$$\frac{\varphi_\infty(\tilde{\omega} = \omega \big| \tilde{s}_i)}{\mathbf{E}_{\varphi_\infty}\left[u'_i(\tilde{\omega}, \underline{w}^\delta_{i,\infty}(\tilde{\omega}))\big|\tilde{s}_i\right]} = \frac{\nabla_{y(\omega)} K(h_\infty, \mathbf{0})}{u'_i(\omega, \underline{w}^\delta_{i,\infty}(\omega))}. \tag{5}$$

Now, observe that if h_∞ is known, then the right hand side of the preceding equation is fixed. For the equality to hold, we infer that the left hand side also cannot depend on the realization of $\tilde{s}_i$. On summing the left hand side over $\omega \in \Omega$, we obtain that $\mathbf{E}_{\varphi_\infty}\left[u'_i(\tilde{\omega}, \underline{w}^\delta_{i,\infty}(\tilde{\omega}))\big|\ \tilde{s}_i\right]$ also does not depend on $\tilde{s}_i$, since the numerator sums to 1. This, in turn, implies that for all $\omega \in \Omega$, $\varphi_\infty(\tilde{\omega} = \omega \big| \tilde{s}_i)$ does not depend on the realization of $\tilde{s}_i$. From this argument we conclude that under the asymptotic common belief φ_∞, the payoff-relevant state $\tilde{\omega}$ is independent of the private signal $\tilde{s}_i$ of each trader i.

By Lemma 4.1, under measure φ_∞, the private signals $\tilde{s}_1, \ldots, \tilde{s}_n$ are independent given the payoff-relevant state $\tilde{\omega}$. Together with Lemma A.1 from Appendix A, this implies that under measure φ_∞, the payoff-relevant state $\tilde{\omega}$ and the joint signal of the traders $\tilde{s} \triangleq (\tilde{s}_1, \ldots, \tilde{s}_n)$ are independent. In other words, we have, almost surely, that

$$\varphi_\infty(\tilde{\omega} = \omega | \tilde{s}) = \varphi_\infty(\tilde{\omega} = \omega).$$

The right hand side is equal to $Q^\delta(\tilde{\omega} = \omega | h_\infty)$. For the left hand side, note that at any time t, we have $\varphi_t(\tilde{\omega} = \omega|\tilde{s}) = \mathbf{P}(\tilde{\omega} = \omega|\tilde{s})$, since the history cannot contain any more information about $\tilde{\omega}$ than the pooled signal $\tilde{s}$. Taking a limit as $t \to \infty$, we obtain that $\varphi_\infty(\tilde{\omega} = \omega|\tilde{s}) = \mathbf{P}(\tilde{\omega} = \omega|\tilde{s})$, thus establishing information aggregation. □

5. Allocative Efficiency and Competitive Equilibrium One of the primary reasons for organizing an information market is to ensure efficient allocation of risk among the participating traders. Moreover, efficiency is important to ensure that traders do not collude among themselves to leave the market, and trade through other mechanisms. In this section, we consider the allocative efficiency of an asymptotically smooth market. We study *ex post* Pareto efficiency, which requires that the set of traders' limiting portfolios be Pareto efficient, if the information of all the traders is revealed.

Definition 5.1 **(Ex Post *Pareto Efficiency*)** *An allocation of securities among the traders is* ex post *Pareto efficient, if holding the total quantity of securities fixed, no trader's expected utility under the pooled information posterior* $\mathbf{P}(\cdot|\tilde{s})$ *can be increased without decreasing some other trader's expected utility.*

Ex post Pareto efficiency is tied closely to the concept of *competitive equilibrium.* Formally, consider an exchange economy among the traders where each trader maximizes expected utility with respect to $\mathbf{P}(\cdot|\tilde{s})$, and trader i has an initial portfolio given by w_i. A *competitive equilibrium* for this economy is specified by an allocation w_i^* for each trader i, and a price $p(\omega)$ for each security ω such that the following two conditions hold:

(i) *Market clearance.* The net portfolio of all the traders in the allocation $\{w_i^* \ : \ 1 \leq i \leq n\}$ should be the same as that in the initial allocation, that is,

$$\sum_{i=1}^n w_i^*(\omega) = \sum_{i=1}^n w_i(\omega), \quad \text{for all } \omega \in \Omega. \tag{6}$$

(ii) *Utility maximization.* For each trader i, the allocation w_i^* of the security to trader i should maximize her expected utility with respect to the pooled information posterior $\mathbf{P}(\cdot|\tilde{s})$, given the price vector p. In other words, for each trader i, the allocation w_i^* is a solution to the optimization problem

$$\begin{array}{ll} \text{maximize} & \mathbf{E}_{\mathbf{P}}\left[u_i(\tilde{\omega}, w_i^*(\tilde{\omega})) \middle| \tilde{s}\right] \\ \text{subject to} & \displaystyle\sum_{\omega \in \Omega} p(\omega) w_i^*(\omega) = \sum_{\omega \in \Omega} p(\omega) w_i(\omega). \end{array} \tag{7}$$

The first fundamental theorem of welfare economics states that the allocation in any competitive equilibrium is *ex post* Pareto optimal. Thus, it suffices for our purposes to show that the limiting allocation of securities among the traders together with the limiting prices in the market constitute a competitive equilibrium under pooled information.

Observe that if information is aggregated, then every trader effectively chooses to stop trading given the pooled private signals $\tilde{s}$; in particular, no trader can profitably improve her expected utility at the limit inferior of her portfolio sequence. Informally, we might therefore expect that the limiting allocation, together with the limiting prices, must be a competitive equilibrium of the economy where traders maximize their expected utility conditional on $\tilde{s}$. The following theorem establishes this fact formally.

Theorem 5.1 *Consider a PBE with strategy profile δ satisfying asymptotic smoothness (Assumption 4.1) and portfolio boundedness (Assumption 4.2). Then, the collection of limit inferiors of the traders' portfolios $\{\underline{w}^{\delta}_{i,\infty} \ : \ 1 \leq i \leq n\}$ and the price vector $\phi \triangleq \nabla_y K(h_\infty, \mathbf{0})$ together constitute a competitive equilibrium of an exchange economy with pooled information, where each trader i's initial portfolio is specified by $\underline{w}^{\delta}_{i,\infty}$.*

Proof. Observe that the market clearance condition (6) holds trivially in this case, since the final and initial allocation are the same. So, in order for the allocation $\{\underline{w}^{\delta}_{i,\infty} \ : \ 1 \leq i \leq n\}$ and the price vector ϕ to constitute a competitive equilibrium, we require that the utility maximization condition holds for all traders. As the maximization problem (7) is a convex program, sufficient conditions for an allocation $\{w_i : \ 1 \leq i \leq n\}$ to be optimal can be written as

$$\mathbf{P}\left(\tilde{\omega} = \omega \middle| \tilde{s}\right) u_i'(\omega, w_i(\omega)) = \lambda p(\omega), \quad \text{for all } \omega \in \Omega,$$

where $\lambda \in \mathbb{R}$ is a Lagrange multiplier.

Now we know from Theorem 4.1 that the market aggregates the information of the traders. Thus, we obtain $\mathbf{P}\left(\tilde{\omega} = \omega \middle| \tilde{s}\right) = \varphi_\infty(\tilde{\omega} = \omega) = \varphi_\infty(\tilde{\omega} = \omega \middle| \tilde{s}_i)$. Then, the sufficiency conditions become

$$\varphi_\infty(\tilde{\omega} = \omega \middle| \tilde{s}_i) u_i'(\omega, w_i(\omega)) = \lambda p(\omega).$$

It is now easily seen from (5) that the sufficiency conditions are satisfied by the allocation $w_i \triangleq \underline{w}^{\delta}_{i,\infty}$ for the price vector $p \triangleq \phi \triangleq \nabla_y K(h_\infty, \mathbf{0})$, by setting the Lagrange multiplier $\lambda \triangleq \mathbf{E}_{\varphi_\infty}\left[u_i'(\tilde{\omega}, \underline{w}^{\delta}_{i,\infty}(\tilde{\omega}))\right]$. This implies that the allocation $\underline{w}^{\delta}_{i,\infty}$ is utility maximizing for trader i with the price vector ϕ. □

Suppose for a moment that portfolios actually converge, i.e., that for each trader i, $w^{\delta}_{i,T} \to \underline{w}^{\delta}_{i,\infty}$ as $T \to \infty$ and the trader asymptotically ceases trading. The preceding argument proves that the final portfolio of each trader is *ex post* Pareto optimal, in the sense that if the traders were subsequently allowed to trade among themselves circumventing the market maker, they would not have any incentive to do so. In this respect, asymptotically smooth markets exhibit allocative efficiency.

We conclude with a brief discussion regarding the information content of prices. Let $\phi \triangleq \nabla_y K(h_\infty, \mathbf{0})$ denote the eventual price vector in a PBE with strategy profile δ satisfying asymptotic smoothness (Assumption 4.1) and portfolio boundedness (Assumption 4.2). Because the limiting allocation is a competitive equilibrium, it is clear that prices must reflect *risk-adjusted* probabilities; this is captured by (5), which reveals that ϕ is the posterior state distribution, shaded by a factor proportional to the marginal utility in each state. In particular, if at least one trader is risk neutral, then ϕ will be equal to the posterior state distribution.

Although prices may not exactly reflect the posterior state distribution in general, by the definition of information aggregation, an observer of the market having the same knowledge as market participants can infer the information contained in the traders' private signals. However, this is too stringent a requirement. We identify a milder requirement under which an observer can identify the posterior distribution of $\tilde{\omega}$ in the following theorem.

THEOREM 5.2 *Consider a PBE with strategy profile δ satisfying asymptotic smoothness (Assumption 4.1) and portfolio boundedness (Assumption 4.2). An observer with access to the final price vector $\phi \triangleq \nabla_y K(h_\infty, \mathbf{0})$ and the limit inferior of the sequence of portfolios of a single trader i, $\underline{w}^\delta_{i,\infty}$, along with her utility function u_i, can infer the posterior distribution φ_∞ of $\tilde{\omega}$.*

PROOF. Note that, as was argued at the end of the proof of Theorem 4.1, under these hypotheses, $\tilde{s}_i$ is independent of $\tilde{\omega}$ under the measure φ_∞. The first order condition (5) in the proof of Theorem 4.1 can be rewritten as, for all $\omega \in \Omega$,

$$\frac{\varphi_\infty(\tilde{\omega} = \omega)}{\mathbf{E}_{\varphi_\infty}\left[u_i'(\tilde{\omega}, \underline{w}^\delta_{i,\infty}(\tilde{\omega}))\right]} = \frac{\nabla_{y(\omega)} K(h_\infty, \mathbf{0})}{u_i'(\omega, \underline{w}^\delta_{i,\infty}(\omega))} \triangleq R_i(\omega),$$

almost surely. We then have, for all $\omega \in \Omega$, almost surely,

$$\varphi_\infty(\tilde{\omega} = \omega) = R_i(\omega) \mathbf{E}_{\varphi_\infty}\left[u_i'(\tilde{\omega}, \underline{w}^\delta_{i,\infty}(\tilde{\omega}))\right].$$

Let $U_i(\omega) = u_i'(\omega, \underline{w}^\delta_{i,\infty}(\omega))$ for all $\omega \in \Omega$. Then the preceding set of equations can be rewritten in vector form as

$$(I_m - R_i U_i^\top)\varphi_\infty = \mathbf{0},$$

where I_m is the $m \times m$ identity matrix, and we view R_i and U_i as column vectors.

The result then follows from observing that the matrix $I_m - R_i U_i^\top$ has rank $m-1$, which implies that φ_∞ is uniquely determined as the probability vector in the null space of the aforementioned matrix. Thus, an uninformed observer with access to R_i and U_i can infer the common belief φ_∞. □

6. Cost Function Market Maker In this section we focus our attention on a class of algorithmic market makers defined via *cost functions*. In such a market, the price seen by a trader is set by a fixed rule that depends only on the total outstanding number of shares sold in the market. Cost function market makers encompass a wide class of markets, and of particular significance is the fact that market scoring rules can be reformulated as cost functions [1, 3]. The ease of organizing a market based on a cost function has led to its use in many real settings, especially in combinatorial prediction markets. Our main observation is that for such markets, the asymptotic smoothness condition developed in Section 4 can be replaced by an assumption of finite loss to the market maker to ensure information aggregation. This latter assumption is shown to be readily satisfied by many well-studied cost functions. Thus, in many markets based on cost functions, portfolio boundedness suffices for information aggregation.

Let $q_t(\omega)$ denote the total quantity of the security ω sold by the market maker until time t. We have the following relation between q_t and the trades $\{y_\tau \,:\, 1 \leq \tau \leq t\}$ that have occurred up to time t:

$$q_t = \sum_{\tau=1}^{t} y_\tau.$$

A cost function market maker is defined by a continuously differentiable function $C\colon \mathbb{R}^m \to \mathbb{R}$, satisfying the following condition:

$$C(q + a\mathbf{1}) = C(q) + a, \text{ for all } a \in \mathbb{R}. \tag{8}$$

After any history h_t, the market maker prices the trade $y \in \mathbb{R}^m$ at time t according to

$$K(h_t, y) \triangleq C(q_{t-1} + y) - C(q_{t-1}).$$

Market makers based on cost functions are a special case of the general class of market makers studied in this paper. These market makers are distinguished by the fact that their pricing function depends on the history only through the total quantity of securities sold up to current time. Further, observe that the total revenue obtained by the market maker at time t is given by

$$\sum_{\tau=1}^{t} K(h_\tau, y_\tau) = \sum_{\tau=1}^{t} C(q_\tau) - C(q_{\tau-1}) = C(q_t) - C(\mathbf{0}).$$

Thus, at any time, the total revenue of the market maker is also dependent only on the total quantity of securities sold, and not on the actual sequence of trades leading to the final position. Finally, note that the condition imposed by (8) on the cost function ensures that the traders cannot profit via an exchange of money with the market maker. This implies that a cost function market maker prices the trades considering only the risk involved in the trade.

The loss incurred by the market maker is given by the net change in the portfolio of all the traders. Thus, the loss of the market maker at time t, if the payoff-relevant state ultimately realized is $\tilde{\omega} = \omega$, is given by $\sum_{i=1}^{n} (w_{i,t}(\omega) - w_{i,0}(\omega))$. The next assumption we make requires that in the PBE, the market maker's loss is almost surely finite for every payoff-relevant state.

Assumption 6.1 ***(Finite Loss)*** *For the PBE with strategy profile δ, there holds*

$$\limsup_{t\to\infty} \sum_{i=1}^{n} w_{i,t}^{\delta}(\omega) < \infty,$$

almost surely for all $\omega \in \Omega$.

The following theorem shows that for cost function market makers, under the portfolio boundedness and finite loss assumptions, the information of the traders is aggregated. Thus in such markets, we do not explicitly require the asymptotic smoothness assumption on the pricing function to obtain information aggregation.

Theorem 6.1 *In any PBE of a cost function market maker satisfying both portfolio boundedness (Assumption 4.2) and finite loss (Assumption 6.1), the market aggregates the information of the traders.*

Before we begin the proof, we first define the projection $\Gamma\colon \mathbb{R}^m \to \mathbb{R}^m$ by

$$\Gamma(q) \triangleq q - \left(\frac{1}{m} \sum_{\omega\in\Omega} q(\omega)\right) \mathbf{1}.$$

For a PBE with strategy profile δ, as $C(q+a\mathbf{1}) = C(q)+a$ for all $q \in \mathbb{R}^m$, we have for all $y \in \mathbb{R}^m$,

$$K(h_t, y) = C(q^{\delta}_{t-1}+y) - \left(\frac{1}{m}\sum_{\omega\in\Omega} q^{\delta}_t(\omega)\right)\mathbf{1} - \left(C(q^{\delta}_{t-1}) - \left(\frac{1}{m}\sum_{\omega\in\Omega} q^{\delta}_t(\omega)\right)\mathbf{1}\right)$$
$$= C(\Gamma(q^{\delta}_{t-1})+y) - C(\Gamma(q^{\delta}_{t-1})).$$

Also, as the total quantity of the securities sold by the market maker at time t equals the sum of the net change in the portfolio of all the traders and the total revenue obtained by the market maker, we have

$$q^{\delta}_t = \sum_{i=1}^{n}\left(w^{\delta}_{i,t} - w^{\delta}_{i,0}\right) + \left(\sum_{\tau=1}^{t} K(h_\tau, y^{\delta}_\tau)\right)\mathbf{1}. \tag{9}$$

This implies that

$$\Gamma(q^{\delta}_t) = \sum_{i=1}^{n}\left(w^{\delta}_{i,t} - w^{\delta}_{i,0} - \left(\frac{1}{m}\sum_{\omega\in\Omega}\left(w^{\delta}_{i,t}(\omega) - w^{\delta}_{i,0}(\omega)\right)\right)\mathbf{1}\right). \tag{10}$$

We use the following lemma in the proof of Theorem 6.1.

Lemma 6.1 *In any PBE δ of the cost-function based market satisfying portfolio boundedness and bounded loss, we have for all $\omega \in \Omega$, almost surely,*

$$\limsup_{t\to\infty} \Gamma(q^{\delta}_t)(\omega) < \infty,$$
$$\liminf_{t\to\infty} \Gamma(q^{\delta}_t)(\omega) < \infty.$$

Proof. From (10), we have for all $\omega \in \Omega$,

$$\Gamma(q^{\delta}_t)(\omega) = \sum_{i=1}^{n} w^{\delta}_{i,t}(\omega) - \frac{1}{m}\sum_{\omega'\in\Omega}\left(\sum_{i=1}^{n} w^{\delta}_{i,t}(\omega')\right) - e(\omega),$$

where $e \triangleq \Gamma\left(\sum_{i=1}^{n} w^{\delta}_{i,0}\right)$. As all the sums involved are finite sums, the result follows easily from the portfolio boundedness and finite loss assumptions. □

Proof of Theorem 6.1. As before, consider a unilateral deviation for trader i after history h_t, such that instead of following strategy δ_i, the trader instead decides to trade a quantity $z \in \mathbb{R}^m$, and stops trading thereafter. The portfolio of the trader after such a deviation is given by

$$w^{z}_{i,t} \triangleq w^{\delta}_{i,t} + z(\omega) - K(h_t, z).$$

As δ is a PBE, such a deviation cannot lead to higher utility for the trader. This implies that, almost surely after any history h_t,

$$\mathbf{E}_{\nu_i(h_t,\tilde{s}_i)}\left[u_i(\tilde{\omega}, w^{z}_{i,t}(\tilde{\omega}))\right] \leq \mathbf{E}_{\nu_i(h_t,\tilde{s}_i)}\left[u_i(\tilde{\omega}, \underline{w}^{\delta}_{i,\infty}(\tilde{\omega}))\right]. \tag{11}$$

Next, let h_∞ be any infinite history that can arise in equilibrium. Consider the time periods when trader i trades in the market under h_∞. From Lemma 6.1, we infer that almost surely, there exists a subsequence $t_k^i : k \geq 1$ (dependent on h_∞), such that along this subsequence, the quantity $\Gamma(q_t^\delta)$ converges to some finite limit, which we denote by q_∞^δ. This implies that along this subsequence, the pricing function $K(h_t, \cdot)$ converges to a limit $G(h_\infty, \cdot)$, given by

$$G(h_\infty, z) \triangleq C(q_\infty^\delta + y) - C(q_\infty^\delta) \text{ for all } z \in \mathbb{R}^m.$$

Note that the limit $G(h_\infty, z)$ is continuously differentiable and $G(h_\infty, \mathbf{0}) = 0$.

Next, we can write the left hand side expression of (11) as

$$\mathbf{E}_{\nu_i(h_t, \tilde{s}_i)} \left[u_i(\tilde{\omega}, w_{i,t}^z(\tilde{\omega})) \right] = \sum_{\omega \in \Omega} \varphi_{i,t}(\tilde{\omega} = \omega) u_i(\omega, w_{i,t}^\delta(\omega) + z(\omega) - K(h_t, \omega)).$$

Taking limits along the subsequence $t_k^i : k \geq 1$, we obtain,

$$\begin{aligned}
&\liminf_{k \to \infty} \mathbf{E}_{\nu_i(h_{t_k^i}, \tilde{s}_i)} \left[u_i(\tilde{\omega}, w_{i,t_k^i}^z(\tilde{\omega})) \right] \\
&= \liminf_{k \to \infty} \sum_{\omega \in \Omega} \varphi_{i,t_k^i}(\tilde{\omega} = \omega) u_i(\omega, w_{i,t_k^i}^\delta(\omega) + z(\omega) - K(h_{t_k^i}, z)) \\
&\geq \sum_{\omega \in \Omega} \varphi_{i,\infty}(\tilde{\omega} = \omega) u_i \left(\omega, \liminf_{k \to \infty} \left(w_{i,t_k^i}^\delta(\omega) + z(\omega) - K(h_{t_k^i}, z) \right) \right) \\
&\geq \sum_{\omega \in \Omega} \varphi_{i,\infty}(\tilde{\omega} = \omega) u_i \left(\omega, \underline{w}_{i,\infty}^\delta(\omega) + z(\omega) - G(h_\infty, z) \right). \qquad (12)
\end{aligned}$$

Moreover, by an argument similar to the proof of Theorem 4.1, we obtain

$$\liminf_{k \to \infty} \mathbf{E}_{\nu_i(h_{t_k^i}, \tilde{s}_i)} \left[u_i(\tilde{\omega}, \underline{w}_{i,\infty}^\delta(\tilde{\omega})) \right] = \mathbf{E}_{\varphi_{i,\infty}} \left[u_i(\tilde{\omega}, \underline{w}_{i,\infty}^\delta(\tilde{\omega})) \right]. \qquad (13)$$

Thus, from (11), (12) and (13), we obtain for all z,

$$\mathbf{E}_{\varphi_{i,\infty}} \left[u_i(\tilde{\omega}, \underline{w}_{i,\infty}^\delta(\tilde{\omega}) + z(\tilde{\omega}) - G(h_\infty, z)) \right] \leq \mathbf{E}_{\varphi_{i,\infty}} \left[u_i(\tilde{\omega}, \underline{w}_{i,\infty}^\delta(\tilde{\omega})) \right].$$

The remainder of the argument follows the proof of Theorem 4.1, with $G(h_\infty, \cdot)$ playing the role of $K(h_\infty, \cdot)$. □

The preceding result implies that for any algorithmic market maker using a cost function, if the market maker suffers at most a finite loss, and portfolio boundedness (Assumption 4.2) holds, then information is aggregated. It is important to note that in almost all cost function markets of interest, the loss of the market maker can be bounded above independent of the sequence of trades. In particular, we now present an assumption on the cost function which ensures that finite loss (Assumption 6.1) holds. Formally, we consider a class of cost function market makers satisfying the following condition.

Assumption 6.2 ***(Bounded Loss Cost Function)***

$$\sup_{q \in \mathbb{R}^m} \left\{ \max_{1 \leq j \leq m} q(j) - C(q) \right\} < \infty.$$

Assumption 6.2 is satisfied by a number of well-studied cost functions considered in the literature, and can be readily verified (see, e.g., [1] for examples). For example, the assumption holds for the cost function C corresponding to the *logarithmic market scoring rule* [8], given by

$$C(q) \triangleq b \log \left(\sum_{\omega \in \Omega} \exp(q(\omega)/b) \right),$$

for some $b > 0$. Similarly, the assumption holds true for the *quadratic market scoring rule* [3], given by

$$C(q) \triangleq \frac{1}{m} \sum_{\omega \in \Omega} q(\omega) + \frac{1}{4b} \sum_{\omega \in \Omega} q(\omega)^2 - \frac{1}{4mb} \left(\sum_{\omega \in \Omega} q(\omega) \right)^2 - \frac{b}{m},$$

for some $b > 0$. Note that Assumption 6.2 imposes a restriction on the model primitives, in particular the cost function, rather than on the equilibrium.

To see that Assumption 6.2 implies finite loss to the market maker, note that from (9), the loss of the market maker at time t is given by:

$$\sum_{i=1}^{n} (w_{i,t} - w_{i,0}) = q_t - \left(\sum_{\tau=1}^{t} K(h_\tau, y_\tau) \right) \mathbf{1} = q_t - (C(q_t) - C(\mathbf{0}))\, \mathbf{1}.$$

It follows from Assumption 6.2 that the loss of the market maker is bounded above uniformly for all t. This in turn implies that finite loss (Assumption 6.1) holds. Thus, in a market based on a bounded loss cost function, *if portfolio boundedness holds, then information is aggregated.*

7. Hierarchically Conditionally Independent Signals Thus far, we considered a model where signals are *conditionally independent*: conditional on the true payoff-relevant state $\tilde{\omega}$, the signals of the traders are independent. Conditionally independent signals are plausible if each trader independently conducts her own research about the true state $\tilde{\omega}$. To achieve conditional independence, we must assume that this process of obtaining information is inherently noisy, with independent noise across traders. Though traders may differ in the accuracy of their signals, a key assumption is that each trader conducts their research in complete isolation: signal information is not shared across traders, except through the market.

As a consequence, this model of obtaining information ignores the possibility that some traders might obtain their information from others in the market. For example, consider a scenario where some subset of traders first carry out research to estimate the state $\tilde{\omega}$. Other traders then contact this “primary” set; for these traders, their signals are inherently correlated with the signals of those primary traders they contacted. For example, an individual investor may obtain information secondhand from the research department of their brokerage. Our signal structure does not allow these correlation relationships, and thus our proofs of information aggregation no longer apply directly.

We refer to these more general correlation relationships as *hierarchical conditional independence.* The central probabilistic construct we require to capture this signal model is a *Markov random field.* A Markov random field captures correlation structure between a collection of random variables through an underlying graph.

Formally, a Markov random field is a probability measure associated with an undirected graph $G = (V, E)$, with vertex set V and edge set E. For any vertex $v \in V$, define $\partial v \triangleq \{u \in V : uv \in E\}$ to be the set of adjacent vertices, and let $\bar{\partial} v \triangleq \partial v \cup \{v\}$. Finally, let $G \backslash S$ denote the graph that remains if the set of nodes S is removed from V, and any edges with endpoints in S are removed from E.

Definition 7.1 *A collection of random variables $\{X_u : u \in V\}$ is a* ***Markov random field*** *with respect to a graph $G = (V, E)$ if, for all $S, A, B \subset V$ where A is disconnected from B in $G \backslash S$, $\{X_v : v \in A\}$ is conditionally independent of $\{X_v : v \in B\}$ given $\{X_v : v \in S\}$.*

We now give the definition of hierarchically conditionally independent signals. For ease of notation, define $\tilde{s}_0 \triangleq \tilde{\omega}$. We have the following definition.

Definition 7.2 *The traders' signals are said to be* ***hierarchically conditionally independent****, if the set of random variables $\{\tilde{s}_0, \ldots, \tilde{s}_n\}$ form a Markov random field with respect to a graph G on the set $\{0, \ldots, n\}$, with no simple cycles containing the node 0.*

The key condition in the preceding definition is that the graph G contains no simple cycles with the node 0. Since $\tilde{s}_0$ is the payoff-relevant state $\tilde{\omega}$, this assumption implies that we can partition traders into subsets as follows. Let $\mathcal{T}_0 = \{1, \ldots, k\}$ be the traders directly connected to node 0 in G; informally, these are the traders who obtain information about the state by directly doing independent research. For each i, let $\mathcal{T}_i$ be the set of traders who obtain their information (directly or indirectly) from trader i (including trader i herself); assume every trader lies in one such set. Hierarchical conditional independence then requires that *the sets $\mathcal{T}_1, \ldots, \mathcal{T}_k$ are mutually disjoint.* In other words, each trader in $\mathcal{T}_0$ is a primary source of information for some subset of other traders; but no trader can have two primary sources.

Hierarchically conditionally independent signals are a generalization of conditionally independent signals. This can be seen at once by noticing that, for conditionally independent signals, the set $\{\tilde{s}_0, \ldots, \tilde{s}_n\}$ forms a Markov random field with respect to a tree on $\{0, \ldots, n\}$, rooted at 0, with an edge $(0, i)$ for all $i \in \{1, \ldots, n\}$. At the same time, hierarchically conditionally independent signals are not fully general: for example, this model precludes a trader obtaining information from two different primary sources, e.g., an online investing site *and* their brokerage.

We now show that our main result continues to hold if the traders' signals are hierarchically conditionally independent. We have the following extension of Theorem 4.1, proving information aggregation with hierarchically conditionally independent signals in a complete market.

Theorem 7.1 *In any PBE of a complete market with hierarchically conditionally independent signals, if asymptotic smoothness (Assumption 4.1) and portfolio boundedness (Assumption 4.2) hold, then the market aggregates the information of the traders.*

Proof. Note that the proof for Theorem 4.1 uses conditional independence of the signals only via Lemma 4.1 and Lemma A.1 from Appendix A. Thus, to extend the proof, we need counterparts of those lemmas for hierarchically conditionally independent signals. The counterpart for Lemma 4.1 follows. The proof is deferred to Appendix A.

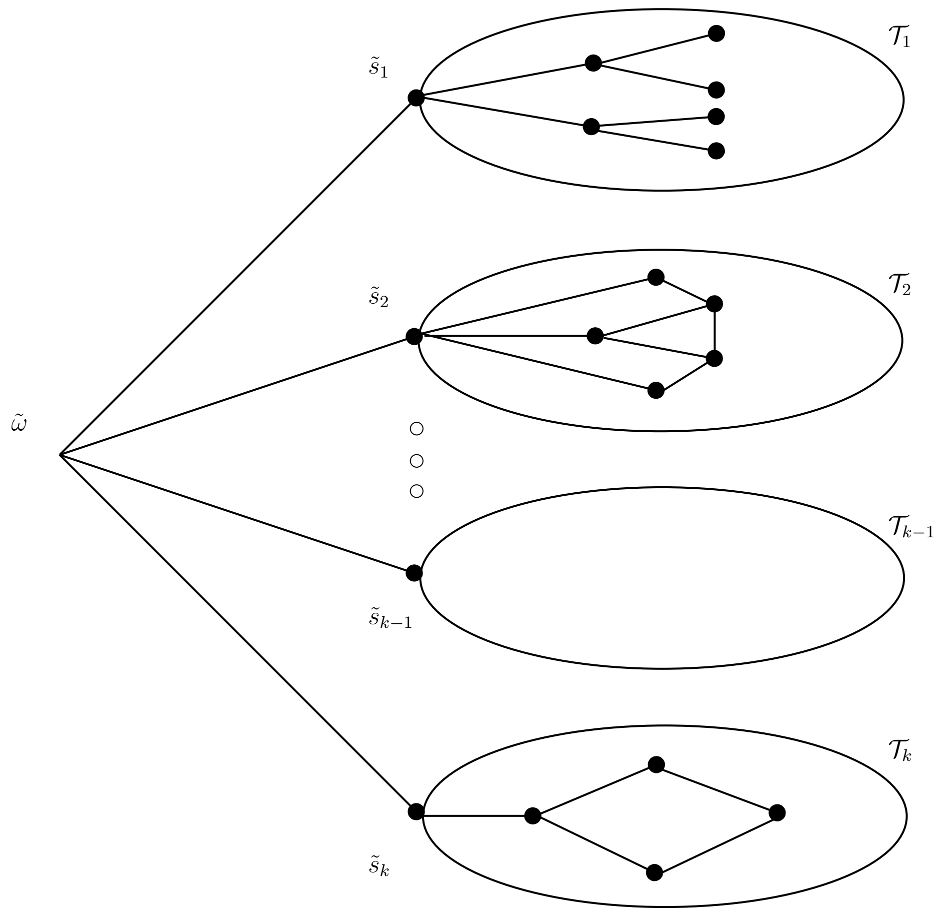

Figure 1: Hierarchically conditionally independent signals

LEMMA 7.1 *Suppose that the traders' signals are hierarchically conditionally independent with respect to a graph G before the start of trading. Then in any PBE with strategy profile δ, under Q^δ, conditional on the history h_t, the traders' signals $\tilde{s}_1, \dots, \tilde{s}_n$ are hierarchically conditionally independent with respect to the same graph G for any $1 \leq t \leq \infty$.*

The counterpart for Lemma A.1 is Lemma A.2 in Appendix A. □

Can the signal structure be generalized further, and still yield information aggregation? To partly investigate this question, we end this section with an example of a market where the traders' signals have a more general dependence: we show that no trade is a PBE in this market, thus proving that information aggregation may not occur with stronger correlation in the signals of traders.

EXAMPLE 7.1 *Let $\Omega = \{0,1\}$, and let the state $\tilde{\omega}$ be distributed uniformly at random. The signal space for each trader is given by $\Sigma_i = \{0,1\}$. The signals of the traders are distributed according to the following distribution:*

$$\mathbf{P}(\tilde{s} = (0,0)|\ \tilde{\omega} = 0) = \mathbf{P}(\tilde{s} = (1,1)|\ \tilde{\omega} = 0) = 0.5,$$
$$\mathbf{P}(\tilde{s} = (0,1)|\ \tilde{\omega} = 1) = \mathbf{P}(\tilde{s} = (1,0)|\ \tilde{\omega} = 1) = 0.5.$$

That is, if the true state $\tilde{\omega} = 0$, then the traders' signals are the same and equal to 0 or 1 with equal probability. On the other hand, if the true state $\tilde{\omega} = 1$, then the traders' signals are again 0 or 1 with equal probability, but now the signal are complementary to each other. Note that by pooling their information, the two traders can always exactly determine the true state $\tilde{\omega}$.

Let the market maker use the cost function corresponding to the logarithmic market scoring rule to price the trades. We now present a system of beliefs and a strategy for each trader, which together constitutes a PBE. After any history h, let each trader's belief about the state $\tilde{\omega}$ and the other trader's signal be given by the distribution $\mathbf{P}$ conditioned on their private signal. Note that this belief structure does not condition on the observed history, and thus this prevents any signaling between the traders. Consider a strategy δ^i for the trader i, such that at any time t after any history h, the trade specified by δ^i is given by $y_t = -q_t$, where q_t denotes the total quantity of securities sold by the market maker until time t. Given the belief of a trader about the state $\tilde{\omega}$ and the other trader's future actions, it can be shown that the optimal action for a trader to take is to always bring the total quantity of securities sold by the market maker to zero. Since the actions of the traders do not depend on their signals, the beliefs satisfy the Bayes rule. Thus, it follows that the strategy profile δ and the belief structure form a PBE.

Looking at the equilibrium path for the above PBE, one obtains that there is no trade occurring in the market, if the market maker starts with $q_0 = \mathbf{0}$. Thus at all time periods, the belief of the traders regarding the true state $\tilde{\omega}$ assigns equal probability to both possible alternatives. However, if the signals were to be pooled, then the true state would be known with complete certainty. Thus we observe that, with general signal structure, there exist PBE where information aggregation does not occur.

The preceding example is closely related to an example presented by Ostrovsky (see [14], Example 1), where he shows the importance of separability of the security for information aggregation in his market model. The main distinction is that in our model we have two securities, while in Ostrovsky's example trade is over one security (his security is a linear combination of those in the preceding example).

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Appendix A. Proofs. Proof of Lemma 4.1. The proof follows directly from Lemma 7.1 and the observation that conditionally independent signals are hierarchically conditionally independent with respect to the graph $G = (V, E)$, where $V = \{0, \ldots, n\}$ and $E = \{(0, i) : 1 \leq i \leq n\}$. □

Proof of Lemma 7.1. Let G be the graph with no simple cycles involving node 0, under which the traders' signals are hierarchically conditionally independent under Q^δ (without conditioning on h_t). Recall that $\tilde{s}_0 = \tilde{\omega}$ by definition.

The proof of the lemma is by induction. The base case for $t = 1$ is trivial. Assume the claim holds for $t = T$, i.e. conditioning on the history h_T, the set of random variables $\{\tilde{s}_0, \ldots, \tilde{s}_n\}$ form a Markov random field with respect to G. This implies that for every set $S, A, B \subset \{0, \ldots, n\}$ with A and B disconnected in $G \backslash S$, $\{\tilde{s}_i : i \in A\}$ is conditionally independent of $\{\tilde{s}_i : i \in B\}$, given $\{\tilde{s}_i : i \in S\}$ and h_T. Fix one such S, A and B.

Let trader k arrive at the market at time T, i.e. $i_T = k$, and let y_T denote the trade conducted by the trader. We have the following three cases, depending on whether $k \in S$, $k \in A$ or $k \in B$.

(i) $k \in S$: Since y_T depends only on h_T and $\tilde{s}_k$ and $k \in S$, conditioning on $h_{T+1} = (h_T, y_T)$ and $\{\tilde{s}_i : i \in S\}$ is same as conditioning on h_T and $\{\tilde{s}_i : i \in S\}$. Thus it follows trivially that the induction step holds.

(ii) $k \in A$: Again, since y_T depends only on $\tilde{s}_k$ and h_T, we see that conditional on h_T and $\{\tilde{s}_i : i \in S\}$, the trade y_T depends only on the set $\{\tilde{s}_i : i \in A\}$. Thus, further conditioning on y_T cannot change the independence of $\{\tilde{s}_i : i \in A\}$ and $\{\tilde{s}_i : i \in B\}$. Thus, again the induction step holds.

(iii) $k \in B$: The argument is similar as that for $k \in A$.

Thus, regardless of which trader arrives at the market, the induction step carries through. As S, A and B were arbitrary, this proves the result for all $t < \infty$. The result for $t = \infty$ follows through the use of Lemma 3.1. □

We use the following lemmas in the proof of Theorem 4.1 and Theorem 7.1 respectively.

Lemma A.1 *Let X and $\{Y_i\ ; i = 1, \ldots, n\}$ be random variables such that: (1) conditional on X, the random variables $\{Y_i\}$ are independent; and (2) X is independent of Y_i for each i. Then X and $(Y_1, \ldots, Y_n)$ are independent.*

Proof. Let g and $f_1, \ldots, f_n$ be bounded functions. As $Y_1, \ldots, Y_n$ are independent conditional on X, we have

$$\mathbf{E}\left[\prod_i f_i(Y_i) \big| X\right] = \prod_i \mathbf{E}\left[f_i(Y_i) \big| X\right].$$

Moreover, we know that X and Y_i are independent for all i. This implies that $\mathbf{E}\left[f_i(Y_i) \big| X\right] =$

$\mathbf{E}[f_i(Y_i)]$. Thus we obtain:

$$\mathbf{E}\left[\prod_i f_i(Y_i)\big|X\right] = \prod_i \mathbf{E}[f_i(Y_i)].$$

We then reason as follows:

$$\begin{aligned}\mathbf{E}\left[g(X)\prod_i f_i(Y_i)\right] &= \mathbf{E}\left[g(X)\mathbf{E}\left[\prod_i f_i(Y_i)\big|X\right]\right] \\ &= \mathbf{E}[g(X)]\prod_i \mathbf{E}[f_i(Y_i)].\end{aligned}$$

As this is true for all bounded functions g, $f_1, \ldots, f_n$, we infer that X and $(Y_1, \ldots, Y_n)$ are independent. □

Lemma A.2 *Let $\{X_v : v \in V\}$ be a Markov random field with respect to a connected graph $G = (V, E)$. Let $u \in V$ be such that there exists no simple cycle involving u in G. Moreover, let X_u be independent of X_v for each $v \in \partial u$. Then, X_u is independent of $\{X_v : v \in V, v \neq u\}$.*

Proof. Let A_v denote an event for each $v \in V$. As $\{X_v : v \in V\}$ is a Markov random field, we get

$$\begin{aligned}&\mathbf{P}(X_v \in A_v\ \forall v \notin \bar{\partial} u,\ X_u \in A_u \big|\ X_i\ \forall i \in \partial u) \\ &= \mathbf{P}(X_v \in A_v\ \forall v \notin \bar{\partial} u \big|\ X_i\ \forall i \in \partial u) \cdot \mathbf{P}(X_u \in A_u \big|\ X_i\ \forall i \in \partial u).\end{aligned}$$

Next, note that conditional on X_u, the random variables in the set $\{X_v : v \in \partial u\}$ are independent of each other. Furthermore, we have that X_u is independent of X_v for each $v \in \partial u$. Thus, from Lemma A.1, we obtain that X_u is independent of $\{X_v : v \in \partial u\}$. This implies that $\mathbf{P}(X_u \in A_u \big| X_i\ \forall i \in \partial u) = \mathbf{P}(X_u \in A_u)$. Thus, we get

$$\begin{aligned}&\mathbf{P}(X_v \in A_v\ \forall v \notin \bar{\partial} u,\ X_u \in A_u \big| X_i\ \forall i \in \partial u) \\ &= \mathbf{P}(X_v \in A_v\ \forall v \notin \bar{\partial} u \big| X_i\ \forall i \in \partial u) \cdot \mathbf{P}(X_u \in A_u).\end{aligned}$$

This implies by definition that,

$$\mathbf{P}(X_v \in A_v\ \forall v \in V) = \mathbf{P}(X_v \in A_v\ \forall v \neq u) \cdot \mathbf{P}(X_u \in A_u).$$

As this is true for all events A_v, we obtain that X_u is independent of $\{X_v : v \in V, v \neq u\}$. □

Appendix B. Measure Theoretic Formulation. Recall that the history at time t is given by the trades until time t :

$$h_t = (y_1, \ldots, y_{t-1}).$$

Moreover, the infinite history is defined as the entire sequence of trade which takes place in the market, given by $h_\infty \triangleq (y_1, \ldots, y_t, \ldots,)$. Let H_t denote the set of all possible histories till time t, and let H_∞ denote the set of all possible infinite history. We endow H_∞ with the topology of pointwise convergence, and let $\mathcal{G}_\infty$ denote the Borel σ-algebra on H_∞.

Let $(\Sigma_i, \mathcal{T}_i)$ be a measure space, and let $\mathcal{T}$ denote the product σ-algebra on Σ. Moreover, let $\mathcal{C}$ denote the complete σ-algebra on Ω. Finally, let $\mathcal{S} \triangleq H_\infty \times \Omega \times \Sigma$, and let $\mathcal{F}_\infty$ denote the product σ-algebra on $\mathcal{S}$.

Note that $(\mathcal{S}, \mathcal{F}_\infty)$ captures all the uncertainty in our model. In particular, the random variables $\tilde{\omega}$ and $\tilde{s}_1, \ldots, \tilde{s}_n$ are all measurable with respect to the measure space $(\mathcal{S}, \mathcal{F}_\infty)$.

The joint prior distribution $\mathbf{P}$ over the state $\tilde{\omega}$ and the signals $\tilde{s}_1, \ldots, \tilde{s}_n$ shared by all the traders and the market maker can now be represented as a measure over the restricted probability space $\big(\Omega \times \Sigma, \sigma(\mathcal{C} \times \mathcal{T})\big)$.

Beliefs. The belief $\nu_i(h_t, s_i)$ of a trader i after each history h_t, and having observed signal s_i, is represented as a probability measure on the set $(\mathcal{S}, \mathcal{F}_\infty)$.

Strategies. The strategy δ_i of a trader i can be represented as a set of random variables $\delta_i(h_t, s_i)$ such that $\delta_i(h_t, \tilde{s}_i)$ is measurable with respect to the sub-σ-algebra generated by h_t and $\tilde{s}_i$. Note that if δ_i is a mixed strategy, then we have to consider a larger measure space to include the randomization of the trader.

Given the joint prior $\mathbf{P}$ on $\tilde{\omega}$ and the signals $\tilde{s}_1, \ldots, \tilde{s}_n$, a PBE with strategy profile δ and the belief ν_i for each trader i induces a probability measure Q^δ on $(\mathcal{S}, \mathcal{F}_\infty)$ through the Kolmogorov extension theorem. The measures $\varphi_{i,t}$ and φ_t can then be defined as in the main text.